

- Notice that arcs  $CE$  and  $DE$  are also quarter-circles. Thus,  $\triangle OCE \cong \triangle ODE \cong \triangle OAB$ , so triangle  $CED$  has double the area of triangle  $AOB$ . The answer is **(E)**  $2 : 1$ .
- Let the circle have center  $O$ , and let the tangent intersect the circle at point  $T$ . Then  $AT = \frac{4}{3}r$  and  $OT = r$ . Using the Pythagorean theorem, we have  $AO = \sqrt{AT^2 + OT^2} = \frac{5}{3}r$ . The shortest distance from  $A$  to the circle is simply the length of  $AO$  minus the radius of the circle, or  $\frac{5}{3}r - r = \frac{2}{3}r$ . Since  $l = \frac{4}{3}r$ , this is also equal to **(C)**  $\frac{1}{2}l$ .
- By the extended law of sines, we have  $2R = \frac{12}{\sin 30^\circ} = 24$ .  $2R$  is the length of the diameter, so the answer is **(C)**  $24$ .
- Since  $AB \perp BC$ , we have  $AO = \sqrt{AB^2 + BO^2} = \sqrt{(2r)^2 + r^2} = \sqrt{5}r$  by the Pythagorean theorem. Then  $AD = AO - OD = \sqrt{5}r - r = (\sqrt{5} - 1)r$ . We also have  $PB = AB - AP = AB - AD = 2r - (\sqrt{5} - 1)r = (3 - \sqrt{5})r$ . Checking choice (A), we see that  $AP^2 = (\sqrt{5} - 1)^2 r^2 = (6 - 2\sqrt{5})r^2$ , and that  $PB \cdot AB = (3 - \sqrt{5})r \cdot 2r = (6 - 2\sqrt{5})r^2$ . Thus, the answer is **(A)**  $AP^2 = PB \cdot AB$ .
- Let the triangle have side lengths  $6 + 8, 6 + x, 8 + x$ . Notice that the semiperimeter of this triangle is  $14 + x$ . Then by Heron's,  $K = \sqrt{(14 + x) \cdot x \cdot 8 \cdot 6} = rs = 4(14 + x)$ . Squaring both sides, this is equivalent to  $48(14 + x)x = 16(14 + x)^2$ , and dividing by  $16(14 + x)$ , this is  $3x = 14 + x$ , or  $x = 7$ . Thus, the shortest length of this triangle is  $6 + 7 =$  **(B)**  $13$ .
- Denote the angle opposite the side of length 40 as  $\theta$ . By the Law of Cosines, we have  $\cos \theta = \frac{39^2 + 25^2 - 40^2}{2 \cdot 25 \cdot 39} = \frac{546}{2 \cdot 25 \cdot 39} = \frac{7}{25}$ . It follows that  $\sin \theta = \frac{24}{25}$ . By the extended law of sines, we have  $2R = \frac{40}{\sin \theta} = \frac{40}{\frac{24}{25}} =$  **(B)**  $\frac{125}{3}$ .
- (Note: the "mean proportional" refers to the geometric mean.) Notice that the two arcs  $m, n$  sum to the total circumference of 19; that is,  $m + n = 19$ . For  $t$  to be the geometric mean of  $m, n$  and to be an integer, we must have  $mn = m(19 - m)$  be a perfect square. But checking for integer values of  $m$ , we find that this is never the case, so the answer is **(A)** zero.
- Draw segment  $OB$ . Since  $OB = BC$ , we have  $\angle BOC = \angle BCO = y$ . Then,  $\angle ABO = \angle BOC + \angle BCO = 2y$ . Since  $AO = OB$ , we have  $\angle BAO = \angle ABO = 2y$ . Then  $\angle AOD = \angle CAO + \angle ACO = 2y + y = 3y$ , so **(A)**  $x = 3y$ .
- Let triangle  $ABC$  have height  $h$  (opposite side  $BC$ ). Let  $BC$  have midpoint  $M$ . By the Pythagorean theorem,  $BM = \sqrt{12^2 - h^2}$ . Also by the Pythagorean theorem,  $ME = \sqrt{8^2 - h^2}$ . Then  $BE \cdot EC = (\sqrt{12^2 - h^2} + \sqrt{8^2 - h^2})(\sqrt{12^2 - h^2} - \sqrt{8^2 - h^2}) = (12^2 - h^2) - (8^2 - h^2) = 80$ . Then by power of a point,  $80 = AE \cdot ED = 8 \cdot ED$ , so  $ED = 10$ . It follows that  $AD = 8 + 10 =$  **(E)**  $18$ .
- Let the angle trisectors meet  $AB$  at points  $D, E$ , with  $AD < AE$ . We notice that  $\angle CDE = \angle A + 30^\circ$  and  $\angle CED = \angle B + 30^\circ$ . But since  $BC < AC$ , we have  $\angle A < \angle B$ , so  $\angle CDE < \angle CED$ , meaning that  $CE < CD$ . Thus, we search for the value of  $CE$ .



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Problem Set 13.1 - Circles

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We have that  $\sin CEA = \sin(A+60^\circ) = \sin A \cos 60^\circ + \cos A \sin 60^\circ$ . Since  $\sin A = \frac{3}{5}$ ,  $\cos A = \frac{4}{5}$ , this simplifies to  $\frac{3+4\sqrt{3}}{10}$ . Now, by the law of sines on  $ACE$ ,  $\frac{4}{\frac{3+4\sqrt{3}}{10}} = \frac{CE}{\frac{3}{5}}$ , or  $CE = \frac{24}{3+4\sqrt{3}} =$

(A) $\frac{32\sqrt{3} - 24}{13}$
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