Math Olympiad and Problem Solving Programs

G220 - Intermediate Math Olympiad

Problem Set 12.2 - AMC12 and AIME Review

Name: Date:

1. Suppose G and H are the midpoints of AB and BC respectively, and let$O$ be the center of $ABCDEF$. Note that $GHIJKL$ is also regular and shares center O. Observe that GOA is a 30-60-90 right triangle. Since any two regular hexagons are similar, we have $\frac{[ABCDEF]}{[GHIJKL]} = \frac{OA^2}{OG^2} = \frac{4}{3}$. The answer is then $100 \cdot \left(\frac{3}{4}\right)^2 = \boxed{75}$.

2. Let r denote the radius of the third circle. Then the sides of the triangle are $10, 3+r$, and $7+r$. Using Heron's formula and equating this with the given area,we have $84 = \sqrt{(10+r)(r)(7)(3)}$ from which $r = 14, -24$. Since r is positive, it follows that the area of the third circle is $14^2\pi = 196\pi$, so the answer is $\boxed{196}$.

3. Since $AB = BC = CA$, Ptolmey's theorem $AD \cdot BC + AC \cdot BD = AB \cdot CD$ reduces to $AD + BD = CD$. Thus, our answer is $2006 - 2005 = \boxed{1}$.

4. First, we notice that $ac + bd - ad - bc = (a-b)(c-d)$. Now, since $c = \frac{4}{3}a$, we have $\sqrt{a^2+c^2} = \sqrt{a^2 + \frac{16}{9}a^2} = \frac{5}{3}a$. Similarly, $\sqrt{b^2+d^2} = \frac{5}{3}b$. Thus, we have $\frac{5}{3}(a-b) = 15$, or $a - b = 9$. Then $c - d = \frac{4}{3}(a-b) = 12$, so our answer is $9 \cdot 12 = \boxed{108}$.

5. The computer can match man 19 with woman 19 or 20, so there are two choices for man 19. Then, the computer has two choices for man 18 (18, 19, 20 minus the one taken by man 19). Similarly, man 17 will then have two chocies, followed by man 16, and so on all the way to man 1. Thus, there are $\boxed{2^{19}}$ ways.

6. First, we show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{a+c}{b+d}$. To do this, cross multiply the final equation to get $a(b+d) = b(a+c)$, or $ab + ad = ab + bc$, or $ad = bc$. But this is equivalent to $\frac{a}{b} = \frac{c}{d}$, so we are done.

   Now, using this principle, we have $\frac{y}{x-z} = \frac{x+y}{z} = \frac{x+2y}{x}$. Thus, $\frac{x+2y}{x} = \frac{x}{y}$. Let $\frac{x}{y} = u$. Then we have $1 + \frac{2}{u} = u$, or $u + 2 = u^2$. Solving this quadratic gives $u = -1, 2$. Since x, y must both be positive, we have $u > 0$, so $u = \boxed{2}$.

7. The probability of James picking up a fair coin, then flipping 6 heads in a row is $\frac{2}{3} \cdot \frac{1}{2^6} = \frac{1}{96}$. The probability of him picking up the double-headed coin, then flipping 6 heads in a row is $\frac{1}{3}$. Thus, the conditional probability of him having picked up the double-headed coin given that he flipped six head is $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{96}} = \boxed{\textbf{(D)}\ \frac{32}{33}}$.

8. Recall that $\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$.

   We show by induction that $\sum_{n=1}^{k} \arctan \frac{1}{n^2-n+1} = \arctan k$. For the base case $k = 1$, this is clearly true. Now assume it is true for some integer r. Then we have

$$\begin{aligned}\sum_{n=1}^{r+1} \arctan \frac{1}{n^2-n+1} &= \sum_{n=1}^{r} \arctan \frac{1}{n^2-n+1} + \arctan \frac{1}{(r+1)^2-(r+1)+1} \\ &= \arctan r + \arctan \frac{1}{(r+1)^2-(r+1)+1} \\ &= \arctan r + \arctan \frac{1}{r^2+r+1} \\ &= \arctan \frac{r + \frac{1}{r^2+r+1}}{1 - \frac{r}{r^2+r+1}} \\ &= \arctan \frac{r(r^2+r+1)+1}{r^2+r+1-r} \\ &= \arctan \frac{r^3+r^2+r+1}{r^2+1} \\ &= \arctan \frac{(r^2+1)(r+1)}{r^2+1} \\ &= \arctan r+1,\end{aligned}$$

as desired.

Thus, we have $\sum_{n=1}^{\infty} \arctan \frac{1}{n^2-n+1} = \lim_{k\to\infty} \arctan k$, which is $\boxed{\frac{\pi}{2}}$.

9. We find the sum of all possible hundreds digits, then tens digits, then units digits. Every one of $\{1,2,3,4,5,6,7,8,9\}$ may appear as the hundreds digit, and there are $9 \cdot 8 = 72$ choices for the tens and units digits. Thus the sum of the hundreds places is $(1+2+3+\cdots+9)(72)\times 100 = 45 \cdot 72 \cdot 100 = 324000$.

   Every one of $\{0,1,2,3,4,5,6,7,8,9\}$ may appear as the tens digit; however, since 0 does not contribute to this sum, we can ignore it. Then there are 8 choices left for the hundreds digit, and 8 choices afterwards for the units digit (since the units digit may also be 0). Thus, the the sum of the tens digit gives $45 \cdot 64 \cdot 10 = 28800$.

   The same argument applies to the units digit, and the sum of them is $45 \cdot 64 \cdot 1 = 2880$. Then $S = 324000 + 28800 + 2880 = 355\boxed{680}$.

10. We can find a recursion. Let D_n be the number of legal delivery sequences for n houses. Consider any legal sequence of length n. If such a sequence ends with a delivery, we may simply append the delivery to a legal sequence of length $n-1$, of which there are D_{n-1}. If the sequence ends in 1 nondelivery, we append a nondelivery and a delivery to a legal sequence of length $n-2$, of which there are D_{n-2}. If it ends in 2 nondeliveries, we append them and a delivery to all D_{n-3} sequences. Thus,

$$D_n = D_{n-1} + D_{n-2} + D_{n-3}.$$

   Since clearly $D_1 = 2$, $D_2 = 4$, $D_3 = 7$, we can compute the rest: $D_4 = 13$, $D_5 = 24$, $D_6 = 44$, $D_7 = 81$, $D_8 = 149$, $D_9 = 274$, $D_{10} = \boxed{504}$.