Avid Academy
for Gifted Youth

Math Olympiad and Problem Solving Programs

G220 - Intermediate Math Olympiad
Problem Set 8.1 - Roots of Unity and Polynomials
Name: Date:

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1. We know that the 6th roots of unity, except 1, are the roots to $x^5+x^4+x^3+x^2+x+1$; thus, this polynomial is equal to $\frac{x^6-1}{x-1}$. The top factors with difference of squares to $(x^3+1)(x^3-1)$, and these factors each are a sum/difference of cubes, and expanding these out gives $(x+1)(x^2-x+1)(x-1)(x^2+x+1)$. Taking out the $x-1$ factor, we have $x^5+x^4+x^3+x^2+x+1=\boxed{(x+1)(x^2-x+1)(x^2+x+1)}$.

2. Let $P(x)=(1+x)^{2008}=\binom{2008}{0}+\binom{2008}{1}x+\binom{2008}{2}x^2+\cdots+\binom{2008}{2007}x^{2007}+\binom{2008}{2008}x^{2008}$. Plugging in all 4th roots of unity, we have

$$\begin{aligned} P(1)&=\binom{2008}{0}+\binom{2008}{1}+\binom{2008}{2}+\binom{2008}{3}+\cdots\\ P(i)&=\binom{2008}{0}+\binom{2008}{1}i+\binom{2008}{2}i^2+\binom{2008}{3}i^3+\cdots\\ P(-1)&=\binom{2008}{0}+\binom{2008}{1}(-1)+\binom{2008}{2}(-1)^2+\binom{2008}{3}(-1)^3+\cdots\\ P(-i)&=\binom{2008}{0}+\binom{2008}{1}(-i)+\binom{2008}{2}(-i)^2+\binom{2008}{3}(-i)^3+\cdots \end{aligned}$$

Adding these gives us $P(1)+P(i)+P(-1)+P(-i)=$

$$\binom{2008}{0}(4)+\binom{2008}{1}(1+i+-1+-i)+\binom{2008}{2}(1+i^2+(-1)^2+(-i)^2)+\binom{2008}{3}(1+i^3+(-1)^3+(-i)^3)+\cdots$$

We see that for $\binom{2008}{i}$, the coefficient will be 4 if i is divisible by 4, and 0 otherwise. Thus, we have

$$\begin{aligned} \binom{2008}{0}+\binom{2008}{4}+\cdots+\binom{2008}{2008}&=\frac{P(1)+P(i)+P(-1)+P(-i)}{4}\\ &=\frac{(2)^{2008}+(1+i)^{2008}+0^{2008}+(1-i)^{2008}}{4}\\ &=\frac{2^{2008}+2^{1004}+2^{1004}}{4}\\ &=2^{2006}+2^{1003} \end{aligned}$$

3. Since the coefficients of the polynomial are real, the roots come in conjugate pairs. We see that the product of each conjugate pair $z,\overline{z}$ is 1, since $z\overline{z}=|z|^2$, which is 1 since z lies on the unit circle. Thus, the product of all the roots is 1, which makes $d=1$ by Vieta's. Also, since $z\overline{z}=1$, we have $\overline{z}=\frac{1}{z}$. Thus, the reciprocals of the roots of $P(x)$ are the same as the roots of $P(x)$. It then follows by Vieta's that our answer is $\boxed{\textbf{(D)}\ -a}$.

4. The 5th roots of unity are of the form $(1,\frac{2\pi k}{5})$ where k ranges from 0 to 4.

   Comparing the magnitudes of both sides, we must have $|z+2|^5=|z-1|^5$, or $|z+2|=|z-1|$. This gives us that z has a real part of $-\frac{1}{2}$, so let $z=-\frac{1}{2}+bi$. Then we have the equation $(\frac{3}{2}+bi)^5=(-\frac{3}{2}+bi)^5$.

Now let $b = \frac{3}{2}\tan\theta$, so that $\frac{3}{2} + bi = (r, \theta), -\frac{3}{2} + bi = (r, \pi - \theta)$. For their fifth powers to be equal, we must have $(\pi - \theta) - \theta = \frac{2\pi k}{5}$. Rearranging gives $\theta = \frac{5\pi = 2\pi k}{10}$ for $0 \le k \le 4$, or $\theta = \frac{5\pi}{10}, \frac{3\pi}{10}, \frac{1\pi}{10}, -\frac{1\pi}{10}, -\frac{3\pi}{10}$. Note that to have an angle of $\frac{5\pi}{10} = \frac{\pi}{2}$, we must have a real part of 0, contradiction. Thus, we have $\theta = \frac{\pm\pi}{10}, \frac{\pm 3\pi}{10}$, or $b = \pm\frac{3}{2}\tan\frac{\pi}{10}, \pm\frac{3}{2}\tan\frac{3\pi}{10}$. Plugging this back in, we have $z = -\frac{1}{2} \pm \frac{3i}{2}\tan\frac{\pi}{10}, -\frac{1}{2} \pm \frac{3i}{2}\tan\frac{3\pi}{10}$.

5. Let $\omega = \cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}$ be a primitive 15th root of unity. Note that our desired sum is the imaginary part of $\omega + \omega^2 + \ldots + \omega^{14}$. But this sum is equal to -1, so it has an imaginary part of 0; thus, our desired answer is $\boxed{0}$.

6. Representing the numbers in polar form, we have $x = (1, 120^\circ), y = (1, 240^\circ)$. We see that $x^3 = (1, 360^\circ), y^3 = (1, 720^\circ)$; that is, $x^3 = y^3 = 1$. Then we have $x^9 + y^9 = 2$, so $\boxed{\textbf{(C) } x^9 + y^9 = -1}$ must be false.

7. Note that $x^{10} - 1$ has roots $1, \omega, \omega^2, \ldots, \omega^9$. Factoring this with difference of squares, it is equal to $(x^5 - 1)(x^5 + 1)$. Factoring each factor individually, we have that this equals $(x - 1)(x^4 + x^3 + x^2 + x + 1)(x + 1)(x^4 - x^3 + x^2 - x + 1)$. The first factor has root 1, the second has roots $\omega^2, \omega^4, \omega^6, \omega^8$, and the third has root $-1 = \omega^5$. Thus, the last factor have roots $\omega, \omega^3, \omega^7, \omega^9$, and our answer is $\boxed{\textbf{(B) } x^4 - x^3 + x^2 - x + 1 = 0}$.

8. Let $\omega = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1}$ be a primitive $(2n+1)$-th root of unity. We are seeking to prove that the imaginary part of $\omega + \omega^2 + \omega^3 + \ldots + \omega^{2n}$ is 0. But this sum is equal to -1, which has imaginary part 0, so we are done.

9. Elements of the set A are of the form $(1, \frac{2\pi a}{18})$ for integer a, and elements of the set B are of the form $(1, \frac{2\pi b}{48})$ for integer b. We have that the product of these two elements is $(1, \frac{2\pi a}{18} + \frac{2\pi b}{48})$. Dealing with the angle part, we see that this simplifies to $\frac{\pi a}{9} + \frac{\pi b}{24} = \frac{\pi(24a+9b)}{216} = \frac{\pi(8a+3b)}{72}$. Since 8 and 3 are relatively prime, we can pick a, b such that the numerator takes on any integer value. Thus, the angle can be anything of the form $\frac{\pi x}{72} = \frac{2\pi x}{144}$ for integer x. This means that the set C is the set of 144th roots of unity, so our answer is $\boxed{144}$.

10. By the quadratic formula, the solutions are $\frac{1 \pm \sqrt{1 - 4(5i-5)}}{2}$. We have $\sqrt{1 - 4(5i - 5)} = \sqrt{21 - 20i}$. Let $(a + bi)^2 = 21 - 20i$. Expanding the LHS gives us $(a^2 - b^2) + 2abi = 21 - 20i$, and equating real and imaginary parts gives us $a^2 - b^2 = 21, 2ab = -20$. We find the solutions $(a, b) = (5, -2), (-5, 2)$ by inspection. Thus, the solutions to the quadratic are $\frac{1 \pm (5-2i)}{2} = 3 - i, -2 + i$. Our answer is then $3 \cdot -2 = \boxed{\textbf{(B) } -6}$.