Avid Academy
for Gifted Youth

Math Olympiad and Problem Solving Programs

G220 - Intermediate Math Olympiad
Problem Set 7.2 - AMC 12 and AIME Review
Name: Date:

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1. Let $z_1 = 1 + xi = r_1\text{cis }\theta_1$, and let $z_2 = 1 + yi = r_2\text{cis }\theta_2$. Note that $\tan\theta_1 = x, \tan\theta_2 = y$. Then we have $z_1z_2 = (1+xi)(1+yi) = (1-xy)+(x+y)i = (r_1r_2)\text{cis }(\theta_1+\theta_2)$. We have that $\tan(\theta_1+\theta_2) = \frac{x+y}{1-xy}$. Substituting $x = \tan\theta_1, y = \tan\theta_2$, we have $\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$, as desired.

2. We have

$$\begin{aligned}\sin x\cos y + \cos x\sin y &= \frac{(e^{ix}-e^{-ix})(e^{iy}+e^{-iy})}{4i} + \frac{(e^{ix}+e^{-ix})(e^{iy}-e^{-iy})}{4i}\\ &= \frac{(e^{ix+iy}+e^{ix-iy}-e^{iy-ix}-e^{-ix-iy})+(e^{ix+iy}-e^{ix-iy}+e^{iy-ix}-e^{-ix-iy})}{4i}\\ &= \frac{2e^{ix+iy}-2e^{-ix-iy}}{4i}\\ &= \frac{e^{i(x+y)}-e^{-i(x+y)}}{2i}\\ &= \sin(x+y),\end{aligned}$$

as desired.

3. Note that x_1, x_2, x_3, x_4, x_5 are the roots of $z^5 - 1 = 0$. By Vieta's, it immediately follows that $\displaystyle\sum_{1\leq i<j\leq 5} x_ix_j = 0$.

4. Let us first count the numbers less than or equal to 1000 that are not divisible by 2, 3, or 5. (The problems asks for numbers strictly less than 1000, but we can include 1000 since it is clearly divisible by 2.)

   By the Principle of Inclusion-Exclusion, the amount of numbers less than or equal to 1000 which are divisible by 2, 3, or 5 is $\lfloor\frac{1000}{2}\rfloor + \lfloor\frac{1000}{3}\rfloor + \lfloor\frac{1000}{5}\rfloor - \lfloor\frac{1000}{6}\rfloor - \lfloor\frac{1000}{10}\rfloor - \lfloor\frac{1000}{15}\rfloor + \lfloor\frac{1000}{30}\rfloor = 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$. Thus, there are $1000 - 734 = 266$ numbers less than 1000 not divisible by 2, 3, or 5. Note that $168 - 3 = 165$ of these are prime (168 primes, but $2, 3, 5$ have already been excluded), and that the number 1 is also not composite. So we subtract 166 away from this count to get that there are $\boxed{\textbf{(A) } 100}$ prime-looking numbers.

5. Taking both sides of the first equation to the a-th power, we have $b = a^{c^{2005}}$. Suppose $c \geq 2$. Then we have $b = a^{c^{2005}} \geq 2^{2^{2005}}$, contradiction. Thus, we must have $c = 0, 1$. If $c = 0$, then we have $b = 1$, after which it follows that $a = 2004$. If $c = 1$, then we have $b = a$, which gives us $a = b = 1002$. We have two total solutions: $(a, b, c) = (2004, 1, 0), (1002, 1002, 1)$, and our answer is $\boxed{\textbf{(C) } 2}$.

6. Note that log_bx is defined only for positive x. Then we have $\log_{2003}\log_{2002}\log_{2001} x > 0$. Taking each side to the power of 2003, we have $\log_{2002}\log_{2001} x > 2003^0 = 1$. Taking each side to the power of 2002, we have $\log_{2001} x > 2002$. Taking each side to the power of 2001, we have $x > 2001^{2002}$, or $\boxed{\textbf{(B)}}$.

7. We have $P(0) = e = 0$. Then our polynomial is equal to $x(x^4 + ax^3 + bx^2 + cx + d)$. Suppose that $d = 0$. Then we would have another factor of x dividing $P(x)$, which would mean that

our polynomial would not have 5 distinct real roots. Contradiction. Thus, our answer is $\boxed{\textbf{(D)}\ d}$.

8. Let the polynomial be $P(x) = (x-r)(x-s)(x^2+ax+b)$, where r, s are the two integer zeroes mentioned in the problem, and a, b are integers. For an answer choice z to be correct, we must have that $z, \overline{z}$ are the roots to the polynomial x^2+ax+b. In particular, we must have $b = z\overline{z}$ be an integer. Checking choice A, we have $\frac{(1+i\sqrt{11})(1-i\sqrt{11})}{4} = 3$. Computing the product similarly for the other choices yields $\frac{1}{2}, \frac{5}{4}, \frac{5}{4}, \frac{7}{2}$, respectively. Thus, our answer must be $\boxed{\textbf{(A)}\ \dfrac{1+i\sqrt{11}}{2}}$.

9. We have $f_1(11) = (1+1)^2 = 4, f_2(11) = 16,\ f_3(11) = (1+6)^2 = 49, f_4(11) = (4+9)^2 = 169$, $f_5(11) = (1+6+9)^2 = 256, f_6(11) = (2+5+6)^2 = 169$. From here, it is clear that the sequence repeats with period 2, so that $f_{1998}(11) = f_6(11) = \boxed{169}$.

10. We have $a+2 = b+1 = c = d-1 = e-2$. We are given that $b+c+d = (c-1)+c+(c+1) = 3c$ is a perfect square, and that $a+b+c+d+e = (c-2)+(c-1)+c+(c+1)+(c+2) = 5c$ is a perfect cube. Let $c = 3^a5^bj$, where j is relatively prime to 3 and 5. We have that $3^{a+1}5^bj$ is a perfect square and $3^a5^{b+1}j$ is a perfect cube. Since each factor is relatively prime, we may consider them separately:

    We must have $a+1$ be even for 3^{a+1} to be a square, and a to be divisible by 3 for 3^a to be a cube. The smallest possible value of a for this is $a = 3$. We must have b be even and $b+1$ be divisible by 3. The smallest b where this is possible is $b = 2$. Finally, we must have j be both a perfect square and cube. The smallest value for which this is possible is $j = 1$. Thus, our smallest possible value for c is $3^35^2 = \boxed{675}$.