



Math Olympiad and Problem Solving Programs
G220 - Intermediate Math Olympiad
Problem Set 7.1 - Roots of Unity

Name:

Date:

1. Note that $z_1 = 2\text{cis } 120$, $z_2 = 8\text{cis } -30$, so $z_1 z_2 = 16\text{cis } 90 = 16i$ and $\frac{z_1}{z_2} = \frac{\text{cis } 150}{4} = -\frac{\sqrt{3}+1}{8}$.
2. We have that $-1 + \sqrt{3}i = 2\text{cis } 120$. Taking this to the 12th power gives $2^{12}\text{cis } 1440 = 2^{12}\text{cis } 0 = 2^{12}$.
3. The 7th roots of unity are of the form $(1, \frac{2\pi k}{7})$, where k ranges from 0 to 6.
4. We have $z = -2 + 2i = (2\sqrt{2}, \frac{3\pi}{4})$. By de Moivre's, the three cube roots are $(\sqrt{2}, \frac{\pi}{4})$, $(\sqrt{2}, \frac{\pi}{4} + \frac{2\pi}{3})$, $(\sqrt{2}, \frac{\pi}{4} + \frac{4\pi}{3})$, which are $(\sqrt{2}, \frac{\pi}{4})$, $(\sqrt{2}, \frac{11\pi}{12})$, $(\sqrt{2}, \frac{19\pi}{12})$.
6. Since the coefficients of the polynomial are real, the roots come in conjugate pairs. We see that the product of each conjugate pair z, \bar{z} is 1, since $z\bar{z} = |z|^2$, which is 1 since z lies on the unit circle. Thus, the product of all the roots is 1, which makes $d = 1$ by Vieta's. Also, since $z\bar{z} = 1$, we have $\bar{z} = \frac{1}{z}$. Thus, the reciprocals of the roots of $P(x)$ are the same as the roots of $P(x)$. It then follows by Vieta's that our answer is **(D) $-a$** .
7. We have $(1+i)^2 = 1+2i+i^2 = 2i$, and $(1-i)^2 = 1-2i+i^2 = -2i$. Our answer is then $(2i)^{1004} - (-2i)^{1004}$, which clearly cancels out to **(C) 0**.
8. The 5th roots of unity are of the form $(1, \frac{2\pi k}{5})$ where k ranges from 0 to 4.
Comparing the magnitudes of both sides, we must have $|z+2|^5 = |z-1|^5$, or $|z+2| = |z-1|$. This gives us that z has a real part of $-\frac{1}{2}$, so let $z = -\frac{1}{2} + bi$. Then we have the equation $(\frac{3}{2} + bi)^5 = (-\frac{3}{2} + bi)^5$.
Now let $b = \frac{3}{2} \tan \theta$, so that $\frac{3}{2} + bi = (r, \theta)$, $-\frac{3}{2} + bi = (r, \pi - \theta)$. For their fifth powers to be equal, we must have $(\pi - \theta) - \theta = \frac{2\pi k}{5}$. Rearranging gives $\theta = \frac{5\pi - 2\pi k}{10}$ for $0 \leq k \leq 4$, or $\theta = \frac{5\pi}{10}, \frac{3\pi}{10}, \frac{1\pi}{10}, -\frac{1\pi}{10}, -\frac{3\pi}{10}$. Note that to have an angle of $\frac{5\pi}{10} = \frac{\pi}{2}$, we must have a real part of 0, contradiction. Thus, we have $\theta = \frac{\pm\pi}{10}, \frac{\pm 3\pi}{10}$, or $b = \pm \frac{3}{2} \tan \frac{\pi}{10}, \pm \frac{3}{2} \tan \frac{3\pi}{10}$. Plugging this back in, we have $z = -\frac{1}{2} \pm \frac{3i}{2} \tan \frac{\pi}{10}, -\frac{1}{2} \pm \frac{3i}{2} \tan \frac{3\pi}{10}$.