Avid Academy
for Gifted Youth

Math Olympiad and Problem Solving Programs

G220 - Intermediate Math Olympiad

Problem Set 6.1 - Complex Numbers

Name: Date:

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1. For the sake of brevity, we will denote $x = \cos\theta, y = \sin\theta$. Note that $\cos 3\theta + i\sin 3\theta = (x+iy)^3$ by de Moivre's. We expand out the right side to obtain $x^3 + 3x^2yi - 3xy^2 - y^3i$. Equating the imaginary part gives $i\sin 3\theta = 3x^2yi - y^3i$. Dividing through by i and replacing $x^2 = 1 - y^2$, we have $\sin 3\theta = 3y(1-y^2) - y^3 = 3y - 4y^3 = \boxed{3\sin\theta - 4\sin^3\theta}$.

2. We have $|\frac{8-6i}{12+5i}| = \frac{|8-6i|}{|12+5i|} = \boxed{\frac{10}{13}}$.

3. We have $(1+i)^4(2-2i)^3 = (\sqrt{2}\text{cis}\frac{\pi}{4})^4(2\sqrt{2}\text{cis}\frac{-\pi}{4})^3 = 64\sqrt{2}\text{cis}\frac{\pi}{4} = \boxed{64 + 64i}$.

4. We represent the points as $(4,3), (5,-2)$. The line between these two points has slope $\frac{3-{}^-2}{4-5} = \boxed{-5}$.

5. Represent the given points as $(1,2), (-2,1), (-1,-2)$. From drawing the points, it is clear that the other point of the square is $(2,-1)$. Alternatively, we may note that the circumcenter of three given points is the origin, and to make a square, we must have the last point be diametrically opposite from another point. $(1,2)$ and $(-1,-2)$ are already diametrically opposite, so the only other choice is the point opposite $(-2,1)$, which is $(2,-1)$. Either way, we arive at the answer $\boxed{\textbf{(B)}\ 2-i}$.

6. We have $z_2 = 0^2 + i = i$, then $z_3 = i^2 + i = -1 + i$, then $z_4 = (-1+i)^2 + i = -i$, then $z_5 = (-i)^2 + 1 = -1 + i$. It is clear that we will have the pattern repeat between $-i$ and $-1+i$ after this. Since 2005 is odd, we have $z_{2005} = z_3 = -1 + i$, so $|z_{2005}| = \boxed{\textbf{(B)}\ \sqrt{2}}$.

7. $|z|$ is real, so equating the imaginary parts, we find $z = a + 8i$ for some real a. Then we have $a + \sqrt{a^2 + 64} = 2 \iff \sqrt{a^2+64} = 2 - a$. Squaring gives $a^2 + 64 = a^2 - 4a + 4 \iff a = -15$. Thus, $z = -15 + 8i, |z|^2 = \boxed{\textbf{(E)}\ 289}$.

8.

   Solution 1. Let $z = a + 164i$. Multiply through to get $a + 164i = (a + 164i + n)4i = 4ai - 164\cdot 4 + 4ni$. Equating real parts gives $a = -164\cdot 4 = -41\cdot 16$. Equating imaginary parts gives $164i = 4ai + 4ni$. Dividing this by $4i$, we have $n = 41 - a = 41 - (-16\cdot 41) = 17\cdot 41 = \boxed{697}$.

   Solution 2. Let A be the affix of z on the complex plane, and let B be the affix (point represent by) of $z + n$, and let O be the origin. The initial conditions tell us that A, B lie on the line $y = 164$, and that $AO \perp BO, AO = 4BO$. We are looking for the length AB. Let P be the foot of O onto segment AB. We see that $OP = 164$. Then, by similar triangles, $\frac{AO}{BO} = 4 = \frac{PA}{OP} = \frac{OP}{PB}$, implying that $PA = 4\cdot 164, PB = \frac{164}{4}$. From here, we find that $AB = AP + PB = 16\cdot 41 + 41 = \boxed{697}$.

9. We have $(9+bi)^2 = 81 + 18bi - b^2$, $(9+bi)^3 = 729 + 243bi - 27b^2 - b^3i$. Equating the two numbers' imaginary parts, we have $18b = 243b - b^3$, and dividing by b gives $b^2 = 225$, or $b = \boxed{15}$.

10. Using the formula for geometric series, $z^6 + z^3 + 1 = \frac{x^9-1}{x^3-1}$. This expression is 0 at the 9th roots of unity that are not cubic roots of unity; that is, they are of the form $\text{cis}40^\circ \cdot k$, where k is not divisible by 3. The only valid value of k meeting this is $4\cdot 40 = \boxed{160}$.