1. Let $S = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$. Then, multiplying by $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}$, we have

$$\begin{aligned}\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}S &= (\sin\frac{\pi}{7}\cos\frac{\pi}{7})(\sin\frac{2\pi}{7}\cos\frac{2\pi}{7})(\sin\frac{3\pi}{7}\cos\frac{3\pi}{7})\\ &= (\frac{1}{2}\sin\frac{2\pi}{7})(\frac{1}{2}\sin\frac{4\pi}{7})(\frac{1}{2}\sin\frac{6\pi}{7})\\ &= \frac{1}{8}(\sin\frac{2\pi}{7})(\sin\frac{3\pi}{7})(\sin\frac{\pi}{7}).\end{aligned}$$

Dividing back by $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}$, we have that $S = \boxed{\frac{1}{8}}$.

2. (a) We have

$$\begin{aligned}\cos(\theta+\varphi)+\cos(\theta-\varphi) &= (\cos\theta\cos\varphi-\sin\theta\sin\varphi)+(\cos\theta\cos\varphi+\sin\theta\sin\varphi)\\ &= 2\cos\theta\cos\varphi.\end{aligned}$$

Dividing by 2 gives the desired result.

(b) We have

$$\begin{aligned}\cos(\theta-\varphi)-\cos(\theta+\varphi) &= (\cos\theta\cos\varphi+\sin\theta\sin\varphi)-(\cos\theta\cos\varphi-\sin\theta\sin\varphi)\\ &= 2\sin\theta\sin\varphi.\end{aligned}$$

Dividing by 2 gives the desired result.

(c) We have

$$\begin{aligned}\sin(\theta+\varphi)+\sin(\theta-\varphi) &= (\sin\theta\cos\varphi+\cos\theta\sin\varphi)+(\sin\theta\cos\varphi-\cos\theta\sin\varphi)\\ &= 2\sin\theta\cos\varphi.\end{aligned}$$

Dividing by 2 gives the desired result.

3. For the following solutions, let $\frac{\theta+\varphi}{2} = a, \frac{\theta-\varphi}{2} = b$, so that $\theta = a+b, \varphi = a-b$.

(a) Using our substitution, this identity is equivalent to

$$\sin(a+b)+\sin(a-b) = 2\sin a\cos b.$$

Dividing by two, we see that this is the product-to-sum identity 2c, so we are done.

(b) Substitute $\varphi' = -\varphi$. The identity becomes

$$\sin\theta+\sin\varphi' = 2\sin\left(\frac{\theta+\varphi'}{2}\right)\cos\left(\frac{\theta-\varphi'}{2}\right),$$

which is equivalent to 3a.

(c) This identity is equivalent to

$$\cos(a+b)+\cos(a-b) = 2\cos a\cos b.$$

Dividing by two, we see that this is the product-to-sum identity 2a, so we are done.

(d) Using our substitution and multiplying by -1, this inequality is equivalent to

$$\cos(a-b) - \cos(a+b) = 2\sin a \sin b$$

Dividing by two, we see that this is the product-to-sum identity 2b, so we are done.

4. (a) Multiplying both sides by $\sin\frac{\alpha}{2}$, we have that the left hand side is equal to $\sum_{i=0}^{n}\sin(\varphi+i\alpha)\sin\frac{\alpha}{2}$. Applying the product-to-sum identity, we have that this is equal to $\frac{1}{2}\sum_{i=0}^{n}\left(\cos\left(\varphi+i\alpha-\frac{\alpha}{2}\right)-\cos\left(\varphi+i\alpha+\frac{\alpha}{2}\right)\right)$. This sum telescopes (that is, the terms in the middle cancel each other out), and we find that this sum is equal to $\frac{1}{2}\left(\cos\left(\varphi-\frac{\alpha}{2}\right)-\cos\left(\varphi+n\alpha+\frac{\alpha}{2}\right)\right)$. Applying the sum-to-product identity, we have that this is equal to

$$\frac{1}{2}\left(-2\sin\left(\frac{(\varphi-\frac{\alpha}{2})+(\varphi+n\alpha+\frac{\alpha}{2})}{2}\right)\sin\left(\frac{(\varphi-\frac{\alpha}{2})-(\varphi+n\alpha+\frac{\alpha}{2})}{2}\right)\right)$$

Simplifying this, we have

$$-\sin\left(\frac{2\varphi+n\alpha}{2}\right)\sin\left(\frac{-(n+1)\alpha}{2}\right),$$

or

$$\sin\left(\varphi+\frac{n\alpha}{2}\right)\sin\left(\frac{(n+1)\alpha}{2}\right),$$

which was what we wanted.

(b) We proceed similarly to 4a, multiplying over by $\sin\frac{\alpha}{2}$. Using product-to-sum, the LHS becomes $\frac{1}{2}\sum_{i=0}^{n}\left(\sin\left(\frac{\alpha}{2}+(\varphi+i\alpha)\right)-\sin\left(-\left(\frac{\alpha}{2}-(\varphi+i\alpha)\right)\right)\right)$. This telescopes to $\frac{1}{2}\left(\sin\left(\varphi+n\alpha+\frac{\alpha}{2}\right)-\sin\left(\varphi-\frac{\alpha}{2}\right)\right)$. Using sum-to-product, this is

$$\sin\left(\frac{(\varphi+n\alpha+\frac{\alpha}{2})-(\varphi-\frac{\alpha}{2})}{2}\right)\cos\left(\frac{(\varphi+n\alpha+\frac{\alpha}{2})+(\varphi-\frac{\alpha}{2})}{2}\right),$$

which simplifies to

$$\sin\left(\frac{(n+1)\alpha}{2}\right)\cos\left(\varphi+\frac{n\alpha}{2}\right),$$

as desired.

5. $A$ is a point on a circle of radius 1 centered at the origin, and $B$ is a point on a circle of radius 2 centered at the origin. Then we have $AB \geq AO+OB = 3$ by the triangle inequality. Since equality is met, $A, O, B$ must be collinear in that order, so $A = (\cos(75+180), \sin(75+180)) = (\cos 225, \sin 225)$. Since $\sin x = \cos(90-x)$ and $\cos x = \sin(90-x)$, we have that $A = (\sin(90-225), \cos(90-225)) = (\sin -135, \cos -135)$. Adding 360 to each angle, we have $A = (\sin 225, \cos 225)$, so $\boxed{\theta = 225^\circ}$.

6. Note that 1,2,3,5,7,9 can all be in the units place (2 and 5 must be by themselves, and 1 and 9 must have a tens digit). We can then place 4,6,8 into the tens place. It's easy to find such a construction ($\{61, 2, 43, 5, 7, 89\}$), so our answer is $(1+2+3+5+7+9)+10(4+6+8) = \boxed{\textbf{(A) } 207}$.

7. We have $A\log_{200} 5 + B\log_{200} 2 = \log_{200} 5^A 2^B = C$. Taking 200 to the power of both sides, we have $5^A 2^B = 200^C = 5^{2C} 2^{3C}$. Thus, $A = 2C, B = 3C$. We must have $C = 1$ for $\gcd(A, C) = \gcd(B, C) = 1$, so $A = 2, B = 3, C = 1$, giving an answer of $\boxed{\textbf{(A) } 6}$.

8. Multiplying both sides by 7!, we obtain

$$5 \cdot 6! = 3600 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3a_2 + 7 \cdot 6 \cdot 5 \cdot 4a_3 + 7 \cdot 6 \cdot 5a_4 + 7 \cdot 6a_5 + 7a_6 + a_7.$$

Taking this equation mod 7, we have $2 \equiv a_7 \pmod 7$. Since $0 \le a_7 < 7$, it follows that $a_7 = 2$. We now subtract 2 from our equation and divide by 7 to get

$$514 = 6 \cdot 5 \cdot 4 \cdot 3a_2 + 6 \cdot 5 \cdot 4a_3 + 6 \cdot 5a_4 + 6a_5 + a_6.$$

Taking this equation mod 7, we have $4 \equiv a_6 \pmod 6$. It follows that $a_6 = 4$.

Proceeding similarly, we find $a_5 = 0, a_4 = 1, a_3 = 0, a_2 = 1$, so our answer is $2+4+0+1+0+1$, or $\boxed{\textbf{(A) } 8}$.

9. Let such a square be $n^2$. We have that $4 \cdot 6 | n^2$, which implies that $4^2 \cdot 6^2 | n^2$, or $12|n$. Also, we have $n^2 < 10^6$, and taking the square root of both sides, $n < 1000$. Thus, our answer is equal to the number of multiples of 12 under 1000, which is $\left\lfloor \frac{999}{12} \right\rfloor = \boxed{83}$.

10. Note that $\tan(\arctan x + \arctan y) = \frac{x+y}{1-xy}$, so $\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$. Applying this formula, we have $\arctan \frac{1}{3} + \arctan \frac{1}{4} = \arctan \frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\frac{1}{4}} = \arctan \frac{7}{11}$. Proceeding similarly, we have $\arctan \frac{7}{11} + \arctan \frac{1}{5} = \arctan \frac{23}{24}$. Now, we have $\arctan \frac{23}{24} + \arctan \frac{1}{n} = \arctan \frac{23n+24}{24n-23} = \frac{\pi}{4}$. Taking the tangent of both sides, we have $\frac{23n+24}{24n-23} = 1$. Multiplying over gives $23n + 24 = 24n - 23$, which reduces to $n = \boxed{47}$.