Avid Academy
for Gifted Youth

# Math Olympiad and Problem Solving Programs

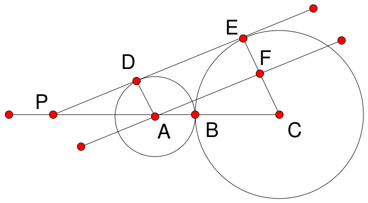
G220 - Intermediate Math Olympiad
Problem Set 5.1 - Cyclic Quadrilaterals
Name: Date:

1. Since $ABCD$ is cyclic, we have that $\angle ADB = \angle ACB = 50^\circ$, and that $\angle ADC = 180^\circ - \angle ABC = 70^\circ$. Then $\angle BDC = \angle ADC - \angle ADB = 70^\circ - 50^\circ = \boxed{20^\circ}$.

2. Looking at triangles $ADH$ and $BCF$, we have that $\frac{1}{2}\angle A + \frac{1}{2}\angle D + \angle DHA = \frac{1}{2}\angle B + \frac{1}{2}\angle C + \angle BFC = 180^\circ$. Adding these up, we have $\frac{1}{2}(\angle A + \angle B + \angle C + \angle D) + \angle DHA + \angle BFC - 360^\circ$. Since the angles of quadrilateral $ABCD$ add up to 360 degrees, we have that $\angle DHA + \angle BFC = \angle EHG + \angle EFG = 180^\circ$, so $EFGH$ is cyclic.

   Now, if we let $\angle FEH = \theta$, then $\angle EFG = 193^\circ - \theta, \angle EHG = 180^\circ - \angle EFG = \theta - 13^\circ$, and $\angle DEC = \theta$. Now, looking at triangles $ADH$ and $CDE$, we have $\frac{1}{2}\angle A + \frac{1}{2}\angle D + (\theta - 13^\circ) = \frac{1}{2}\angle C + \frac{1}{2}\angle D + \theta$, which reduces to $\angle A - \angle C = \boxed{26^\circ}$.

3. In the picture, let the two circles have centers $A, C$. Let one of the external common tangents intersect the two circles at $D, E$, respectively, and let $\overline{DE}$ meet line $\overline{AC}$ at $P$ (where thew two common external tangents will meet). Draw a line perpendicular to $\overline{DE}$ through $A$, intersecting $\overline{CE}$ at $F$. Note that $AC = 1 + 4 = 5, FC = 4 - 1 = 3$, so $\sin\angle FAC = \frac{3}{5}, \cos\angle FAC = \frac{4}{5}$. Since $\angle FAC = \angle EPC$, our answer is $\sin 2\angle FAC = 2(\frac{3}{5})(\frac{4}{5}) = \boxed{\dfrac{24}{25}}$.



4. If two angles have equal sines but unequal cosines, then they must be supplementary. Since this quadrilateral is not cyclic, these supplementary angles are adjacent, and must be a nonisoceles trapezoid. Since the trapezoid is nonisoceles, we cannot have both legs be 3; thus, we can assume that 3 is a base. The other base cannot be 3, our we could have a parallelogram, but we do not have an opposite pair of equal sides. Thus, we can place the other side of length 3 as a leg.

   Assume the other leg was 8. Let the height of this trapezoid be $h$. Then the bottom base has length $3 + \sqrt{8^2 - h^2} \pm \sqrt{3^2 - h^2}$, where the sign depends on if the leg of length 3 goes towards or away from the other leg. Regardless, this expression is greater than or equal to $3 + \sqrt{8^2 - h^2} - 3$, and since the height can be no more than 3, this is greater than or equal to $\sqrt{8^2 - 3^2} = \sqrt{55}$, which is greater than 4. Contradiction.

   Thus, the length of the other leg is 4, and the base is 8. Letting the height of this trapezoid be $h$, we have $3 + \sqrt{4^2 - h^2} \pm \sqrt{3^2 - h^2} = 8$, or $\sqrt{4^2 - h^2} \pm \sqrt{3^2 - h^2} = 5$. If the sign is negative, then the LHS is greater than or equal to $4 - 3 = 1$, contradiction. Thus, we have $\sqrt{4^2 - h^2} + \sqrt{3^2 - h^2} = 5$. Squaring, we have $4^2 - h^2 + 3^2 - h^2 + 2\sqrt{(4^2 - h^2)(3^2 - h^2)} = 25$, or $\sqrt{(h^4 - 25h^2 + 144)} = h^2$. Squaring again yields $25h^2 = 144$, or $h = \frac{12}{5}$. Our area is then $\frac{3+8}{2} \cdot \frac{12}{5} = \boxed{\dfrac{66}{5}}$.

5. Let $\overline{AE}$ intersect $\overline{BC}$ at $P$. Looking at the right triangle $ABP$, we have $\sin A = \frac{1}{\sqrt{50}}$, so $\cos A = \frac{7}{\sqrt{50}}, \tan A = \frac{1}{7}$. Then we can easily compute $EP = \frac{1}{7}, AE = 7$. Now, we find the

area of the region in the outer square but not in the inner square. This is just $4[ABE]$, since taking the region of $ABE$ and considering the four rotations (of 0, 90, 180, 270 degrees) gives us our desired region. The area of this region is $4 \cdot \frac{1}{2} \cdot 7 \cdot 1 = 14$, and our answer is $\sqrt{50}^2 - 14 = \boxed{\textbf{(C) } 36}$.

6. Draw the line parallel to $\overline{AB}$ through $C$, intersection $\overline{AD}$ at point $E$. We have $DE = 2, \angle DEC = 60^\circ$. Let $EC = x$. By the law of cosines, we have $12^2 = 2^2 + x^2 + 2(2)(x)\cos 60^\circ \iff 140 = x^2 + 2x$. Using the quadratic formula, we find $x = -1 \pm \sqrt{141}$. Discarding the negative solution, we have $x = \sqrt{141} - 1$. Drawing perpendiculars of $\overline{EC}$ from $E$ and $C$, intersecting $\overline{AB}$ at $E', C'$, we find that $AB = AE' + E'C' + C'B = 10\cos 60^\circ + (\sqrt{141} - 1) + 8\cos 60^\circ = \sqrt{141} + 8$. Our answer is then $\boxed{149}$.

7. Since $ABCD$ is cyclic, we have $\angle MAB = \angle MDC, \angle MBA = \angle MCD$, so $\triangle MAB \sim \triangle MBC$. Similarly, $\triangle MAD \sim \triangle MBC$. Now, let $DM = x, BM = 2x$.

   Since $\triangle MAB \sim \triangle MBC$, we have $\frac{6}{8} = \frac{2x}{MC}$, so $MC = \frac{8x}{3}$. Since $\triangle MAD \sim \triangle MBC$, we have $\frac{9}{AD} = \frac{\frac{8x}{3}}{x}$, or $\boxed{AD = \dfrac{27}{8}}$.

8. Let $P$ be the intersection of $\overline{AC}$ and $\overline{BD}$. Let $\angle DAC = \angle DBC = x$, $\angle ABD = B_1$, and $\angle APB = \theta$.

   Applying the formula $[ABC] = 2R^2 \sin A \sin B \sin C$ (see 9b) on triangle $ABD$, we have

   $$[ABD] = 2R^2 \sin A \sin B_1 \sin(\theta - x).$$

   Similarly, we have

   $$\begin{aligned}[CBD] &= 2R^2 \sin(180 - A) \sin x \sin(180 - \theta - B_1) \\ &= 2R^2 \sin A \sin x \sin(\theta + B_1).\end{aligned}$$

   Adding the two yields

   $$\begin{aligned}[ABCD] &= 2R^2 \sin A(\sin B_1 \sin(\theta - x) + \sin x \sin(\theta + B_1)) \\ &= 2R^2 \sin A[(\sin B_1 \sin\theta \cos x - \sin B_1 \cos\theta \sin x) + (\sin x \sin\theta \cos B_1 + \sin x \cos\theta \sin B_1)] \\ &= 2R^2 \sin A[\sin B_1 \sin\theta \cos x + \sin x \sin\theta \cos B_1] \\ &= 2R^2 \sin A[\sin\theta \sin(B_1 + x)] \\ &= 2R^2 \sin A \sin\theta \sin B,\end{aligned}$$

   as desired. ■

9. (a) We have $[ABC] = \frac{1}{2}ab\sin C$. By the extended law of sines, $\frac{c}{\sin C} = 2R \iff \sin C = \frac{c}{2R}$. Plugging this in gives us the result. ■

   (b) Using part (a), we have $[ABC] = \frac{abc}{4R}$. By the extended law of sines, $\frac{a}{\sin A} = 2R$, or $a = 2R\sin A$. Similarly, $b = 2R\sin B, c = 2R\sin C$. Plugging all these in gives us the result. ■

10. For both these problems, let $K$ denote the area of triangle $ABC$.

(a) Multiplying both sides by $R$, we have

$$\begin{aligned} Rr &= \frac{2R^2 \sin A \sin B \sin C}{\sin A + \sin B + \sin C} \\ &= \frac{K}{\sin A + \sin B + \sin C}, \end{aligned}$$

where our second step used the result of 9b. Now, recalling that $K = rs \iff r = \frac{K}{s}$, we substitute to obtain

$$\begin{aligned} \frac{RK}{s} &= \frac{K}{\sin A + \sin B + \sin C} \\ \frac{R}{s} &= \frac{1}{\sin A + \sin B + \sin C} \\ 2R(\sin A + \sin B + \sin C) &= a + b + c \end{aligned}$$

But by the extended law of sines, we have $2R \sin A = a$ and similar relations, so we conclude that the statement is true. ■

(b) Note: this proof is very bad, but it shows the potential power of replacing all expressions with functions of $a, b, c$ and bashing out algebra.

Substitute $r = \frac{K}{s} = \frac{2K}{a+b+c}$, $4R = \frac{abc}{K}$ to obtain

$$\frac{2K}{a+b+c} = \frac{abc}{K} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

Clearing denominators, we have

$$2K^2 = abc(a+b+c) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

Squaring (we can since both sides are positive) gives us

$$4K^4 = a^2b^2c^2(a+b+c)^2 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}.$$

We now have $\sin^2 \frac{A}{2} = \frac{1-\cos A}{2}$. But by the law of cosines, $\cos A = \frac{b^2+c^2-a^2}{2bc}$. Substituting this in, we have

$$\begin{aligned} \sin^2 \frac{A}{2} &= \frac{1 - \frac{b^2+c^2-a^2}{2bc}}{2} \\ &= \frac{a^2 - (b^2 - 2bc + c^2)}{4bc} \\ &= \frac{a^2 - (b-c)^2}{4bc} \\ &= \frac{(a-b+c)(a+b-c)}{4bc}. \end{aligned}$$

We now substitute this and the similar expressions in:

$$\begin{aligned}
4K^4 &= a^2b^2c^2(a+b+c)^2 \cdot \frac{(a-b+c)(a+b-c)}{4bc} \cdot \frac{(b-c+a)(b+c-a)}{4ca} \cdot \frac{(c-a+b)(c+a-b)}{4ab} \\
256K^4 &= (a+b+c)^2(a-b+c)^2(a+b-c)^2(-a+b+c)^2 \\
16K^2 &= (a+b+c)(a-b+c)(a+b-c)(-a+b+c)
\end{aligned}$$

But this is equivalent to Heron's formula:

$$\begin{aligned}
16K^2 &= (2s)(2(s-a))(2(s-b))(2(s-c)) \\
&= (a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c) \\
&= (a+b+c)(-a+b+c)(a-b+c)(a+b-c),
\end{aligned}$$

so we are done. ■