Avid Academy
for Gifted Youth

Math Olympiad and Problem Solving Programs

G220 - Intermediate Math Olympiad

Problem Set 4.2 - AMC 12 and AIME Review Solutions

Name: Date:

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1. By Stewart's theorem, we have $a(d^2+mn) = b^2m+c^2n$. Plugging in, we have $14(d^2+9\cdot 5) = 13^2\cdot 9 + 15^2\cdot 5$. The RHS is equal to 2646, and dividing by 14, we have $d^2+45 = 189$. Subtracting by 45 and taking the square root, we have $d = 12$.

2. Note that since $5 = AM = BM = CM$, M is the circumcenter of $\triangle ABC$, and $\angle A = 90°$. Thus, $b^2+c^2 = 10^2$. Then, we use the angle bisector to compute $BD = \frac{ac}{b+c} = \frac{10c}{b+c}$, $CD = \frac{10b}{b+c}$.

   Using Stewart's formula for angle bisector, we have $d^2 = bc - mn$, or
   $$\frac{24^2\cdot 2}{7^2} = bc - \frac{100bc}{(b+c)^2}.$$
   Note that $(b+c)^2 = b^2+c^2+2bc = 100+2bc$. Now, substituting $x = bc$, we have
   $$\begin{aligned}\frac{24^2\cdot 2}{7^2} &= x - \frac{100x}{100+2x}\\ &= \frac{x(100+2x)-100x}{100+2x}\\ &= \frac{2x^2}{100+2x}\\ &= \frac{x^2}{50+x}\end{aligned}$$
   Multiplying over by $50+x$, we have $x^2 = \frac{24^2\cdot 2}{7^2}x + \frac{24^2\cdot 100}{7^2}$. Using the quadratic formula, we find $x = -\frac{1200}{49}, 48$. Discarding the negative solution (both b and c must be positive), we have $bc = 48$. Also, we have $(b+c)^2 = 100+2\cdot 48 = 196 = 14^2$. Thus, $b+c = 14$. We now easily find that $\{b,c\} = \boxed{\{6,8\}}$.

3. Using the angle bisector theorem, we can calculate $BD = \frac{91}{22}, CD = \frac{70}{11}$. Now, applying the angle bisector case of Stewart's, we have $AD^2 = bc - mn = 10\cdot\frac{13}{2} - \frac{91}{22}\cdot\frac{70}{11}$. Calculating this out, it turns out to be $\frac{4680}{121}$.

   Now, we may consider triangle ADC. Since E is the midpoint of AD, we may apply the median formula: $CE^2 = \frac{2(AC^2+DC^2)-AD^2}{4} = \frac{2(10^2+(\frac{70}{11})^2)-\frac{3470}{121}}{4}$. Simplifying this gives $CE^2 = \boxed{\dfrac{7330}{121}}$.

4. Let the dice rolls be a,b,c, with $1\le a,b,c\le 6$, $a+b=c$. We list out all possible pairs: (1,1,2), (1,2,3), (2,1,3), (1,3,4), (2,2,4), (3,1,4), (1,4,5), (2,3,5), (3,2,5), (4,1,5), (1,5,6), (2,4,6), (3,3,6), (4,2,6), (5,1,6). Of these, there are 15 possibilities total, eight of which contain a 2. Thus, the answer is $\boxed{\textbf{(D)}\ \dfrac{8}{15}}$.

5. Recall that $\frac{1}{\log_a b} = \log_b a$. Thus, the sum can be reexpressed as $\log_{100!}2+\log_{100!}3+\log_{100!}4+\cdots+\log_{100!}100$. But this is equal to $\log_{100!}(2\cdot 3\cdot 4\cdots 100) = \log_{100!}100! = \boxed{\textbf{(C)}\ 1}$.

6. Squaring the first equation, we have $9\sin^2 A + 24\sin A\cos B + 16\cos^2 B = 36$. Squaring the second equation, we have $16\sin^2 B + 24\sin B\cos A + 9\cos^2 A = 1$. Adding the two equations,

we have $9(\sin^2 A+\cos^2 A)+16(\cos^2 B+\sin^2)B+24(\sin A\cos B+\sin B\cos A) = 37$. Subtracting 25 from both sides yields $24(\sin A\cos B+\sin B\cos A = 12$, which is just $\sin(A+B) = \frac{1}{2}$. Then, we have $\sin C = \sin(A+B) = \frac{1}{2}$, so $\angle C$ is either 30 or 150 degrees. If $\angle C$ was 150 degrees, then $\angle A$ would be less than 30 degrees, and we would have $3\sin A+4\cos B \le 3\cdot\frac{1}{2}+4\cdot 1 = \frac{11}{2} < 6$, which is impossible. Thus, we must have $\angle C = \boxed{\textbf{(A) } 30}$.

7. We are looking for the sum of all numbers of the form $\frac{1}{5^a7^b11^c}$, where a, b, c are nonnegative integers. This is equivalent to the product

$$(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots)(1+\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\cdots)(1+\frac{1}{11}+\frac{1}{11^2}+\frac{1}{11^3}+\cdots).$$

Why? Because after expanding this (infinite) product, any number of the form $\frac{1}{5^a7^b11^c}$ will be present (from the choice of $\frac{1}{5^a}, \frac{1}{7^b}, \frac{1}{11^c}$ from each factor, respectively). Also, each term in the expansion represent the reciprocal of a modular number; there are no "extra" terms created.

This product computes to

$$\begin{aligned}\frac{1}{1-\frac{1}{5}}\cdot\frac{1}{1-\frac{1}{7}}\cdot\frac{1}{1-\frac{1}{11}} &= \frac{1}{\frac{4}{5}}\cdot\frac{1}{\frac{6}{7}}\cdot\frac{1}{\frac{10}{11}}\\ &= \frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\\ &= \boxed{\textbf{(E) } \frac{77}{48}}.\end{aligned}$$

8. Completing the square, we see that the equation is equivalent to $(x-7)^2+(y-3)^2 = 64$. Let $a = x-7, b = y-3$. We have $3x+4y = 3a+4b+33$, so we can instead maximize $3a+4b$ given $a^2+b^2 = 64$. Now, graph the circle $a^2+b^2 = 64$, and consider lines of the form $3a+4b = k$. If we drew such a line on the graph, it would have to intersect the circle for that k value to be attainable. We see that the maximum k possible occurs when the line $3a+4b = k$ is tangent to the circle. (Picture the line $3a+4b = k$ moving outward as k increases. It goes farther away from the center of the circle. The last point that it is still touching the circle, it is just barely touching it on one point, the point of tangency.)

Now, consider the value k such that the line is tangent as desired. Our triangle (formed by the x and y axes, and the line) will have legs of length $\frac{k}{4}, \frac{k}{3}$. By the pythagorean theorem, we find that the hypotenuse will have length $\frac{5k}{12}$. Also, the altitude to this side has length 8 (the radius of the circle). Calculating the area in two ways, we have

$$\begin{aligned}\frac{1}{2}\cdot\frac{k}{4}\cdot\frac{k}{3} &= \frac{1}{2}\cdot\frac{5k}{12}\cdot 8\\ \frac{k^2}{24} &= \frac{5k}{3}\\ k &= 40\end{aligned}$$

Adding 33 to this value, we find that the maximum attainable value is $\boxed{\textbf{(B) } 73}$.

9. Subtracting over $30x^2$, we have $3x^2y^2 - 30x^2 + y^2 = 517$. Now, subtracting 10 from each side, we have $3x^2y^2 - 30x^2 + y^2 - 10 = 507$. This factors as $(3x^2+1)(y^2-10) = 507 = 3 \cdot 13^2$. Note that the factor $3x^2 + 1$ is not divisible by 3, so it can only be $1, 13$, or 13^2. If $3x^2 + 1 = 1$, then $y^2 - 10 = 507$, which is impossible since 517 is not a square number. If $3x^2 + 1 = 13^2$, then $x^2 = 56$, which is impossible. Thus, we must have $3x^2 + 1 = 13$ and $y^2 - 10 = 39$. This gives us $x^2 = 4, y^2 = 49$. Our answer is then $3 \cdot 4 \cdot 49 = \boxed{588}$.

10. Let the first column be 3 targets labeled a, the second column be 3 targets labeled b, and the last column be 2 targets labeled c. When the marksman chooses a column, we can write the letter corresponding to the column. When he is done, we will have an arrangement of *aaabbbcc*. Every arrangement is possible, and each corresponds to an order the marksman can shoot in. Thus, our answer is the number of arrangements of *aaabbbcc*, which is simply $\frac{8!}{3! \cdot 3! \cdot 2!} = \boxed{560}$.