

- Let r_1, r_2 be the radii of circles I and II, respectively. We have that $\frac{60}{360}r_1 = \frac{45}{360}r_2$, or $\frac{1}{6}r_1 = \frac{1}{8}r_2 \iff r_1 = \frac{3}{4}r_2$. Squaring this, we have $r_1^2 = \frac{9}{16}r_2^2$, so the ratio of the circles is **(B)** $9 : 16$.
- We have that $K = 2s$, where K is the area, and s is the semiperimeter. But $K = rs$, so we have $r =$ **(A)** 2 .
- We have $\cos \frac{\theta}{2} = \sqrt{1 - \sin^2 \frac{\theta}{2}} = \sqrt{1 - \frac{x-1}{2x}} = \sqrt{\frac{x+1}{2x}}$. Then, $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sqrt{\frac{x-1}{2x}} \cdot \sqrt{\frac{x+1}{2x}} = \frac{\sqrt{x^2-1}}{x}$. Also, we compute $\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1 = \frac{x+1}{x} - 1 = \frac{1}{x}$. Then, $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{\sqrt{x^2-1}}{x}}{\frac{1}{x}} =$ **(E)** $\sqrt{x^2 - 1}$.
- By the Angle Bisector Theorem, we have $A_1C = \frac{3}{2}, A_1B = \frac{5}{2}$. Then, the triangle PQR will have lengths $PQ = \frac{5}{2}, PR = \frac{3}{2}, QR = 2$, which is a triangle similar to the 3-4-5 triangle. Using the Angle Bisector Theorem again, we have $P_1Q = \frac{5}{4}, P_1R = \frac{3}{4}$. Applying the angle bisector version of Stewart's, we have $d^2 = bc - mn$, or $PP_1^2 = \frac{5}{2} \cdot \frac{3}{2} - \frac{5}{4} \cdot \frac{3}{4} = \frac{45}{16}$. Thus, $PP_1 =$ **(B)** $\frac{3\sqrt{5}}{4}$.
- Consider triangle PQR , and let the altitude from R hit \overline{PQ} at point X . Let $\triangle PXR$ be a 8-15-17 triangle with $PX = 8, XR = 15, RP = 17$, and let $\triangle QXR$ be a 15-20-25 triangle with $QX = 20, XR = 15, RP = 25$. We can see that $\cos P = \frac{8}{17}, \sin Q = \frac{3}{5}$. Thus, this triangle is similar to our desired triangle. We could also obtain another triangle by replacing angle Q with an angle of measure $180^\circ - Q$, since the sine of these are equal. We construct this by reflecting Q over X ; however, in this case, this does not lead to another triangle, since the line QR gets flipped over PQ . Now, we have the triangle PQR with lengths 28, 17, 25. Multiplying each side length by 3 gives us a triangle with lengths 84, 51, 75. Since the corresponding height to the side with 84 is 45, we have that the area is $\frac{1}{2} \cdot 84 \cdot 45 =$ **(B)** 1890 .
- Let A be your house, B be Grandma's house, and X be the point on the river you go to. The length you travel is $AX + XB$. Now reflect Grandma's house over the river to point B' . Notice that $XB = XB'$. Thus, the distance you travel is $AX + XB'$, which is less than or equal to AB' by the triangle inequality. It is easy to compute that the east-west distance between A and B' is 12 miles, and that the north-south distance is 5 miles; thus, $AB' = 13$. This distance is achievable (consider the point where AB' and the river intersect, and use that as the point X), so the minimal distance is in fact **(B)** 13 miles.
- Note that $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. We first consider all triplets (x, y, z) of positive integers with $xyz = 2310$. We have 3 places that we can place each prime, and 5 primes total, giving us 3^5 total triplets. We now count triplets with some two variables equal. Clearly, we cannot have $x = y = z$, since 2310 is not a perfect cube. As for the case where two variables are equal, note that all 5 primes we place are distinct. This means that two variables can be equal if and only if they are both 1; thus, we have the 3 triplets $(1, 1, 2310), (1, 2310, 1), (2310, 1, 1)$ that repeat. Sets cannot have repeated numbers, so we discard these, giving us $3^5 - 3 = 240$ total



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triplets. Each of these triplets now has 3 distinct ordered numbers, and to prevent overcount (since sets are unordered), we divide by $3!$. Thus, our final answer is $\frac{240}{3!} = \boxed{\text{(C) } 40}$.

8. Add the three equations to get $x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{22}{3}$. Multiply the three equations to get $x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz + \frac{1}{xyz} = 4 \cdot 1 \cdot \frac{7}{3} = \frac{28}{3}$. Subtracting the first equation we obtained, we have $xyz + \frac{xyz}{2} = 2$. Letting $xyz = u$, we have $u^2 + 1 = 2u \iff (u - 1)^2 = 0$; thus, $xyz = \boxed{\text{(B) } 1}$.
9. Let $a = \tan x, b = \tan y$. We are looking to compute $\tan(x + y) = \frac{a+b}{1-ab}$. We have $a + b = 25$, and $\frac{1}{a} + \frac{1}{b} = 30$. The latter equation reduces to $\frac{a+b}{ab} = 30$, and substituting $a + b = 25$, we have $ab = \frac{5}{6}$. Our answer is then $\frac{a+b}{1-ab} = \frac{25}{1 - \frac{5}{6}} = \frac{25}{\frac{1}{6}} = \boxed{150}$.
10. Expressing these numbers in base three, we have $1, 3, 4, 9, 10, 12, 13, \dots = 1_3, 10_3, 11_3, 100_3, 101_3, 110_3, 111_3, \dots$, we see that these are the numbers with only 0s and 1s in their base 3 expression (since the powers of 3 must be distinct). The 100th term of this sequence is simply the 100th number of 0s and 1s read in base 3, or simply 100 in binary, read in base 3. We have $100 = 1100100_2$, and reading this in base 3, our answer is $1100100_3 = 729 + 243 + 9 = \boxed{981}$.