1. Applying Stewart's, we have $6(4 \cdot 2 + d^2) = 7^2 \cdot 2 + 5^2 \cdot 4$. Dividing by 2 gives $3(8 + d^2) = 49 + 50 = 99$, then dividing by 3 gives $8 + d^2 = 33$, or $\boxed{d = 5}$.

2. Let the triangle be ABC with $AB = 12, AC = 15$, and angle bisector $AD = 10$. By the angle bisector theorem, we can let $BD = 4x, CD = 5x$. Recall that by Stewart's on an angle bisector, we have $d^2 = bc - mn$. Then we have $10^2 = 12 \cdot 15 - 4x \cdot 5x$, which simplifies to $20x^2 = 80$, or $x = 2$. Then the length of the side is $9x = \boxed{18}$.

3. Let M be the midpoint of BC. Since ABC is isoceles, it follows that $AM \perp BC$, so we have $AM^2 = AB^2 - BM^2 = 49 - 1 = 48$, or $AM = 4\sqrt{3}$. Then the area of the triangle is $\frac{1}{2}2 \cdot 4\sqrt{3} = 4\sqrt{3}$. Using the formula $K = \frac{abc}{4R}$, where K is the area of the triangle, and R is the circumradius of the triangle, we have $R = \frac{abc}{4K} = \frac{7^2 \cdot 2}{4 \cdot 4\sqrt{3}} = \boxed{\frac{49\sqrt{3}}{24}}$.

4. Remember that when using Stewart's on an angle bisector, we have $d^2 = bc - mn$. Then it follows that $d^2 = 6 \cdot 8 - 3 \cdot 4 = 36$, or $d = \boxed{6}$.

5. Using Heron's formula, we have that $K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(6)(7)(8)} = 84$. Then, since $K = \frac{1}{2}bh$, we have $84 = \frac{1}{2} \cdot 14h$, or $h = \boxed{12}$. (Alternatively, we may "realize" that an altitude of 12 gives us a 5-12-13 triangle and a 9-12-15 triangle.)

6. Since $\triangle ABC$ is acute, we have that O lies in the interior of $\triangle ABC$. We have $\angle BOC = 2\angle BAC = 120^\circ$. Considering triangle BOC, with $OB = OC = 1$, we can drop the altitude from O to point M on BC and use the properties of a $30^\circ - 60^\circ - 90^\circ$ triangle to find that $BC = \sqrt{3}$, $OM = \frac{1}{2}$. Then, using the formula $K = \frac{abc}{4R}$, we have $K = \frac{1}{2} \cdot \sqrt{3}\sqrt{1}2 = \frac{\sqrt{3}}{4}$, and plugging this in gives $\frac{sqrt3}{4} = \frac{1 \cdot 1 \cdot \sqrt{3}}{4R}$, which simplifies to $R = \boxed{1}$.

7. Using the angle bisector theorem on $\triangle ABE$, we can let $AB = 2x, AE = 3x$. Similarly, using the angle bisector theorem on $\triangle ADC$ shows us that we can let $AD = y, AC = 2y$. Then, Stewart's with angle bisector ($d^2 = bc - mn$) on triangle ABE gives us $y^2 = 6x^2 - 6$. Similarly, Stewart's on triangle ADC gives us $9x^2 = 2y^2 - 18$. Substituting for y^2 in the second equation, we have $9x^2 = 2(6x^2 - 6) - 18 \iff 3x^2 = 30 \iff x^2 = 10$. Plugging this back into the first equation gives $y^2 = 6(x^2 - 1) = 54$. Noting that the two other sides are $2x$ and $2y$, and that $x < y$, we find the length of the shortest side to be $\boxed{2\sqrt{10}}$.

8. Using the median formula (derived from stewart's), we have $AD = \frac{\sqrt{2(AB^2+AC^2)-BC^2}}{2} = \frac{\sqrt{2(5^2+8^2)-7^2}}{2} = \boxed{\frac{\sqrt{129}}{2}}$.

9. Let $\angle ACB = \theta$. Using the law of sines on triangle BCD, we have that $\frac{1}{\sin 30^\circ} = \frac{BD}{\sin(180^\circ - \theta)} = \frac{BD}{\sin\theta}$. This reduces to $\sin\theta = \frac{BD}{2}$. Since ABC is a right triangle, we have that $\sin\theta = \frac{1}{AC}$. Thus, we have $\frac{BD}{2} = \frac{1}{AC}$, or $AC \cdot BD = 2$. Let $AC = x, BD = \frac{2}{x}$. Now, using Stewart's, we have
$$(1 + x)(BC^2 + x) = (\frac{2}{x})^2 \cdot x + 1.$$

But by the pythagorean theorem, we have $BC^2 = x^2 - 1$, giving

$$\begin{aligned} (1+x)(x+x^2-1) &= \frac{4}{x} + 1 \\ x^3 + 2x^2 - 1 &= \frac{4}{x} + 1 \\ x^4 + 2x^3 - x &= 4 + x \\ 0 &= x^4 + 2x^3 - 2x - 4 \\ &= x^3(x+2) - 2(x+2) \\ &= (x^3-2)(x+2), \end{aligned}$$

or $x = \boxed{\sqrt[3]{2}}$.

10. Let $\angle APB = \alpha, \angle BPC = \beta, \angle CPD = \gamma, \angle DPA = \theta$. Then we have

$$[ABCD] = \frac{1}{2}(24 \cdot 32 \sin\alpha + 32 \cdot 28 \sin\beta + 28 \cdot 45 \sin\gamma + 45 \cdot 24 \sin\theta).$$

Since $\sin\alpha, \sin\beta, \sin\gamma, \sin\theta \le 1$, we have that

$$\begin{aligned} [ABCD] &\le \frac{1}{2}(24 \cdot 32 + 32 \cdot 28 + 28 \cdot 45 + 45 \cdot 24) \\ &= 2002. \end{aligned}$$

Since equality is met, we must have $\sin\alpha = \sin\beta = \sin\gamma = \sin\theta = 1$, or $\alpha = \beta = \gamma = \theta = 90^\circ$. It then follows from the pythagorean theorem that $AB = \sqrt{PA^2 + PB^2}$, and similar relationships. We then have $AB = 40, BC = 4\sqrt{113}, CD = 53, DA = 51$, giving a perimeter of $144 + 4\sqrt{113} = \boxed{\textbf{(E) } 4(36+\sqrt{113})}$.

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