Math Olympiad and Problem Solving Programs

E230 - Advanced Math Competitions

Problem Set 27.2 - Quadrilaterals

Name: Date:

1. We are given $l = 2w - 1$, and we know $p = 2(l + w)$, so $36 = 2(w + 2w - 1) \Rightarrow 18 = 3w - 1 \Rightarrow w = \frac{19}{3}$, so $l = \frac{35}{3}$. The area is $\frac{19}{3} \cdot \frac{35}{3} = \boxed{\frac{665}{9}}$

2. $\boxed{40}$

3. The area of the parallelogram is $\sqrt{18}$, but the problem was free because it didn't ask a question and it was really hard.

4. Use coordinate geometry. Let point D be the origin, and let AD be lined up on the y-axis and let DC be lined up on the x-axis. The line AE passes through $(0, 3)$ and $(1, 0)$, so the slope is -3 and the y-intercept is 3, so the equation of the line is $y = -3x + 3$. The slope of a perpendicular line is the negative inverse of the slope, and the negative inverse of -3 is $\frac{1}{3}$. The line DF has slope $\frac{1}{3}$ and has y-intercept 0, so the equation of the line for DF is $y = \frac{1}{3}x$. Where do they intersect? Set them equal to each other: $\frac{1}{3}x = -3x + 3 \Rightarrow x = \frac{9}{10}$, so the lines intersect at $(\frac{9}{10}, \frac{3}{10})$. Where does DF intersect with BC? We plug in $x = 3$ into its equation: $y = \frac{1}{3}x = \frac{1}{3} \cdot 3 = 1$. Now we have enough information to solve the problem.

   $[ADE] = \frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}$.

   $[DCF] = \frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}$.

   $[DGE] = \frac{1}{2} \cdot 1 \cdot \frac{3}{10} = \frac{3}{20}$.

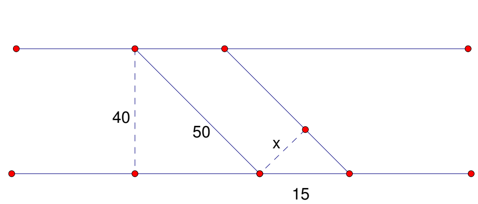
   $[ABFG] = [ABCD] - ([ADE] + [DCF] - [DGE]) = 9 - (\frac{3}{2} + \frac{3}{2} - \frac{3}{20}) = 9 - \frac{57}{20} = \boxed{\frac{123}{20} = 6\frac{3}{20} = 6.15}$

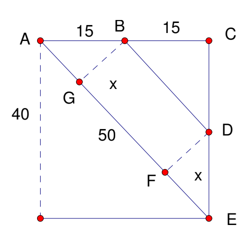
5. $\boxed{38}$

6. If you considered the side lengths and diagonal to be 6, then you would get area $18\sqrt{3}$. This problem was a free point because the question was written incorrectly.

7. $\boxed{50\sqrt{3}}$

8. Here is the picture. The image on the right is a zoom-in of a portion of the sidewalk and the road.

   Notice in the zoom-in that BCD is similar to ACE with a 1:2 ratio, so $CD = DE = 20$. Then $BD = GF = \sqrt{15^2 + 20^2} = 25$. Now we know that $AG + FE = 50 - 25 = 25$. $AG = \sqrt{15^2 - x^2}$ and $FE = \sqrt{20^2 - x^2}$, so we solve:

$$\begin{aligned}
\sqrt{225-x^2}+\sqrt{400-x^2} &= 25 \qquad \text{(square both sides)}\\
225-x^2+2\sqrt{(225-x^2)(400-x^2)}+400-x^2 &= 625\\
625-2x^2+2\sqrt{(225-x^2)(400-x^2)} &= 625\\
\sqrt{(225-x^2)(400-x^2)} &= x^2\\
(225-x^2)(400-x^2) &= x^4\\
90,000-625x^2+x^4 &= x^4\\
90,000 &= 625x^2\\
x &= \boxed{12}
\end{aligned}$$

9. There is a formula for area of a trapezoid with only the side lengths given. If a is the shorter base, b is the larger base, and c and d are the sides, the formula is:

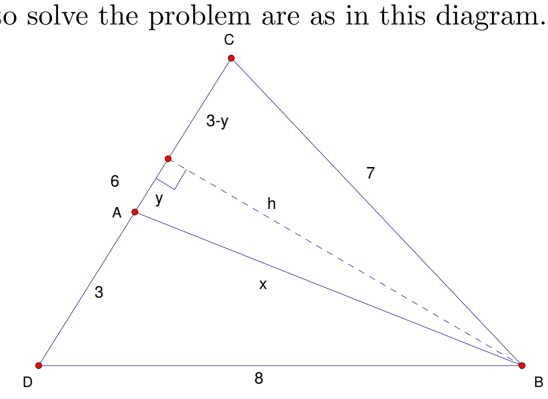
$$\frac{1}{4}\frac{b+a}{b-a}\eta \text{ where } \eta = [(-a+b+c+d)(a-b+c+d)(a-b+c-d)(a-b-c+d)]^{1/2}$$

So first we find η where $a=16, b=30, c=13, d=15$:

$$\begin{aligned}
\eta &= [(-a+b+c+d)(a-b+c+d)(a-b+c-d)(a-b-c+d)]^{1/2}\\
&= [(-16+30+13+15)(16-30+13+15)(16-30+13-15)(16-30-13+15)]^{1/2}\\
&= [42\cdot 14\cdot -16\cdot -12]^{1/2}\\
&= [112896]^{1/2}\\
&= 336
\end{aligned}$$

Then the area is $\frac{1}{4}\frac{30+16}{30-16}\cdot 336 = \frac{46\cdot 336}{4\cdot 14} = \boxed{276}$

10. The variables I use to solve the problem are as in this diagram.



We have three Pythagorean equations:

$$7^2 - (3-y)^2 = h^2 \ (1) \qquad x^2 - y^2 = h^2 \ (2) \qquad 8^2 - (3+y)^2 = h^2 \ (3)$$

Setting (1) and (3) equal to each other and solving for y, we obtain $y = \frac{5}{4}$. Now plugging this value for y into (1) or (3), we obtain $h = \frac{7\sqrt{15}}{4}$. Plugging these values into equation (2), we can obtain x: $x^2 = y^2 + h^2 = \frac{25}{16} + \frac{735}{16} = \frac{760}{16} = \frac{190}{4} \Rightarrow x = \boxed{\frac{\sqrt{190}}{2}}$