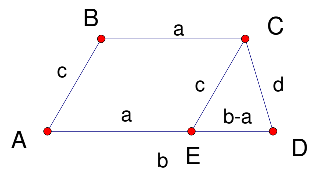
1. First we have to determine which are the parallel sides. Let's consider a trapezoid with an inner parallelogram.



Let's try fitting the side lengths such that $BC = 2$, $AD = 4$, $AB = CE = 1$, and $CD = 3$. We can see that this is impossible, because the triangle inequality says that the sum of any two sides must be greater than the other side, but $1 + 2 = 3$. So the sides 2 and 4 can't be parallel.

Can we come up with a rule to figure out which pairs of parallel sides are possible? Using the variables in the diagram above. From the diagram, $b > a$, we have the following from the Triangle Inequality: $c+d > b-a$,$c+(b-a) > d \Rightarrow d-c < b-a$, $d+(b-a) > c \Rightarrow c-d < b-a$. Combining the last two, we can write $|d - c| < |b - a|$, and we can write the first one as $|b - a| < c + d$, so we can combine these two as $|d - c| < |b - a| < c + d$.

Of the 6 ways to choose a pair of parallel sides, let's see which fit this requirement:

bases		sides					
a	b	c	d	$d-c$	$b-a$	$c+d$	
1	2	3	4	1=	1<	7	no
1	3	2	4	2=	2<	6	no
1	4	2	3	1<	3<	5	yes
2	3	1	4	3>	1<	5	no
2	4	1	3	2=	2<	4	no
3	4	1	2	1=	1<	3	no

There's only one way to place the sides that will work. (That may not always be true!)

So we have $BC = 1$, $AD = 4$, $AB = 2$, $CD = 3$. Now we need to find the height of the triangle CED. Call the height h. We can use the Pythagorean Theorem to find the two parts of the base in terms of h, and then solve for h: $\sqrt{2^2 - h^2} + \sqrt{3^2 - h^2} = 3$. We solve:

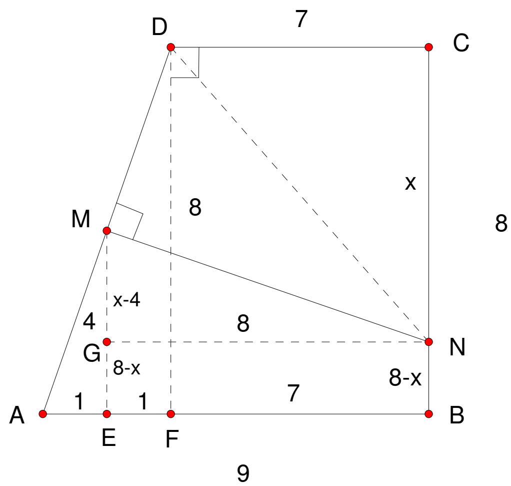
$$\begin{aligned}
\sqrt{4-h^2} + \sqrt{9-h^2} &= 3 \quad \text{(square both sides)} \\
4 - h^2 + 2\sqrt{(4-h^2)(9-h^2)} + 9 - h^2 &= 9 \\
2\sqrt{(4-h^2)(9-h^2)} &= 2h^2 - 4 \\
\sqrt{(4-h^2)(9-h^2)} &= h^2 - 2 \quad \text{(square both sides again)} \\
(4-h^2)(9-h^2) &= (h^2-2)^2 \\
36 - 4h^2 - 9h^2 + h^4 &= h^4 - 4h^2 + 4 \\
32 &= 9h^2 \\
h &= \frac{4\sqrt{2}}{3}
\end{aligned}$$

So the area of the trapezoid is $\frac{1+4}{2} \cdot \frac{4\sqrt{2}}{3} = \boxed{\frac{10\sqrt{2}}{3}}$

2. Draw a line EF parallel to AB and CD, such that F is on CD. Since E is a midpoint, the heights of the four triangles EBC, CEF, AED, and EDF are the same, which we will call h. Also, since E is a midpoint, EF is the average of AD and BC. Call $AD = x$ and $BC = y$. The the area of DEC is $[CEF]+[DEF] = \frac{1}{2} \cdot x + y2 \cdot h + \frac{1}{2} \cdot x + y2 \cdot h = \frac{h(x+y)}{2}$. The problem tells us this area is called S. Now consider the area of the trapezoid, which is the sum of the areas of the triangles.

   $[EBC] + ([CEF] + [AED]) + [EDF] = \frac{1}{2} \cdot y \cdot h + S + \frac{1}{2} \cdot x \cdot h = S + \frac{h}{2}(x+y) = S + S = \boxed{2S}$

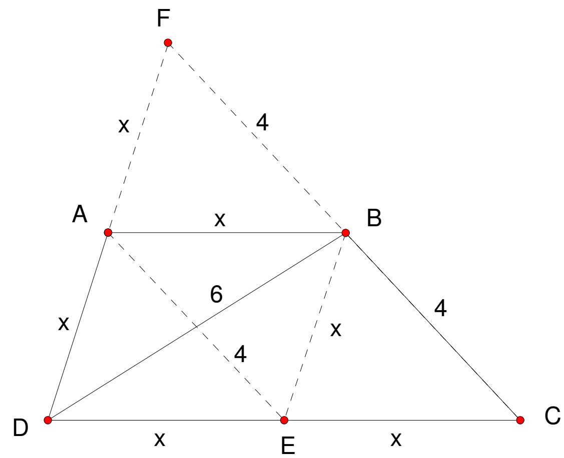
3. Draw the additional lines and points as in the diagram below.



   We determine that $FB = 7$ and $DF = 8$, and since M is a midpoint, $ME = \frac{1}{2}DF = 4$. This means that $\triangle AME \sim \triangle ADF$ with a ratio of 1:2, so then since $AF = AB - FB = 2$, then $AE = EF = 1$. Call $CN = x$, so $NB = GE = 8 - x$, and $MG = 4 - (8 - x) = x - 4$.

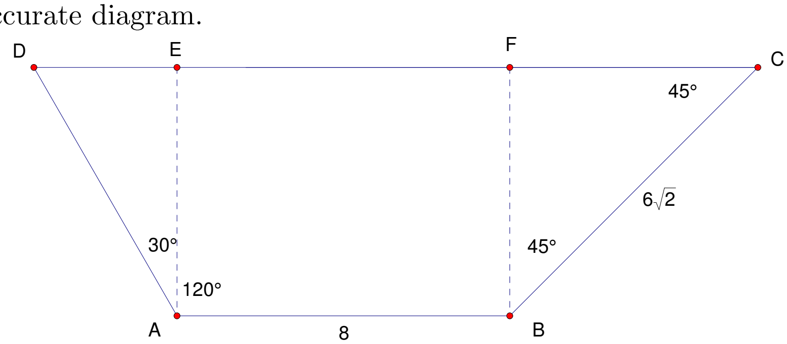
   Now, let's find some right triangles. Since $\triangle MGN$ is a right triangle, $MN = \sqrt{8^2 + (x-4)^2} = \sqrt{x^2 - 8x + 80}$. We can find that $DA = \sqrt{2^2 + 8^2} = 2\sqrt{17}$, so $AM = MD = \sqrt{17}$. Now we can find $DN^2 = 17 + x^2 - 8x + 80 = x^2 - 8x + 97$. Finally, we see that $DC^2 + CN^2 = DN^2$, or $49 + x^2 = x^2 - 8x + 97$, which we solve to obtain $x = 6$. Then $BN = 8 - x = \boxed{2}$

4. Draw a triangle extending from the top of the trapezoid. Also, draw a parallelograms within the trapezoid. See the below diagram.



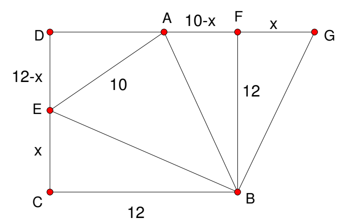
Name $DC = DE = EC = AB = AF = x$. Also, $AE = BC = 4$. Since AB is half the length of DC, AB is a midline and $[FAB] = \frac{1}{4}[FDC]$, so $[ABCD] = 3[FAB]$. Notice that parallelogram $ABED$ is a rhombus with diagonals 4 and 6. The area of a rhombus is half the product of the diagonals, so $[ABED] = \frac{4 \cdot 6}{2} = 12$. So each of the triangles ADE, AEB, BEC, FAB have half this area, or 6. So the area of the trapezoid is $3 \times 6 = \boxed{18}$

5. Draw an accurate diagram.



Now we can see that we have an easy shape with a 30-60-90 triangle, 45-45-90 triangle, and rectangle. We can easily see that since $BC = 6\sqrt{2}$, $BF = 6$. So we have $[AEFB] = 8 \cdot 6 = 48$, $[BCF] = \frac{1}{2} 6 \cdot 6 = 18$. Since AED is 30-60-90, and $AE = 6$, then $DE = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$, so $[AED] = \frac{1}{2} 2\sqrt{3} \cdot 6 = 6\sqrt{3}$. So the area of the trapezoid is $48 + 18 + 6\sqrt{3} = \boxed{66 + 6\sqrt{3}}$

6. Label $CE = x$, then $ED = 12-x$, and by Pythagorean Theorem we have $DA = \sqrt{10^2 - (12-x)^2}$. We draw $\triangle ABG$ that is congruent to $\triangle ABE$. Then draw altitude from B to AG at point F. Then $\triangle ECB \cong \triangle GFB$. So $FG = x$, $FB = 12$, and since $AG = EA = 10$, then $EAF = 10 - x$. See the diagram below.



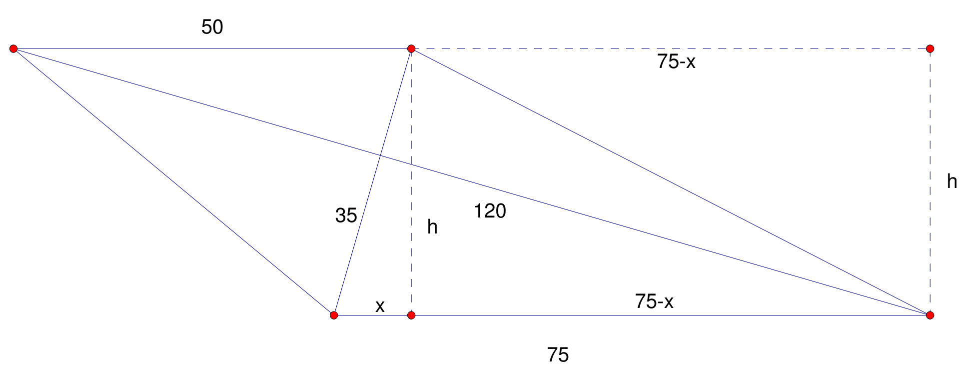
We have $DA = \sqrt{10^2 - (12-x)^2}$, but we also have $DA = DF - AF = 12 - (10-x) = 2 + x$. Now we have an equation.

$$\begin{aligned} 2 + x &= \sqrt{10^2 - (12-x)^2} \\ x^2 + 4x + 4 &= -44 + 24x - x^2 \\ 2x^2 - 20x + 48 &= 0 \\ x^2 - 10x + 24 &= 0 \\ (x-4)(x-6) &= 0 \\ x &= \boxed{4, 6} \end{aligned}$$

7. $\boxed{9}$

8. We know that $ACD = BCD$ because the have the same base and height, so then $AOD = BOC = x$. Also, $[AOB] \cdot [DOC] = [AOD] \cdot [BOC] = 25 \times 4 = 100 = x^2 \Rightarrow x = 10$. So the area of triangles AOD, BOC is 10. So the area of the whole trapezoid is $25 + 4 + 10 + 10 = \boxed{49}$

9. Let us draw an accurate diagram. You can do this by (1) drawing a horizontal line of length 75, (2) drawing a circle with radius 35 with center at the left end of the line from step 1, and drawing a circle with radius 120 at the right end of the line from step 1, (3) find where the the circles can be connected with a horizontal line of length 50.



By drawing some heights and additional lines, we can set up a system of equations with the Pythagorean Theorem. We have $x^2 + h^2 = 35^2$ and $h^2 + (75 - x + 50)^2 = 120^2$. We solve by substituting one expression for h^2: $x^2 + (120^2 - (125 - x)^2) = 35^2$. Solve for x, and we get $x = 9.8$. Then we can find h: $(9.8)^2 + h^2 = 35^2 \Rightarrow h^2 = 1128.96 \Rightarrow h = 33.6$. So the area of the trapezoid is $\frac{50+75}{2} \cdot 33.6 = \boxed{2100}$

10. The trapezoid is an isosceles right trapezoid. Draw a height from B to AD and label the intersection point E. This creates a 45-45-90 triangle, so height $BE = \frac{6}{\sqrt{2}} = 3\sqrt{2}$. Draw a similar height from C to AD and label the intersection point F. Now we have two 45-45-90 triangles, and we know $AE = FD = 3\sqrt{2}$. Let $BC = EF = x$. Now we can find BC by solving the area formula:

$$\begin{aligned}
30 &= \frac{BC + AD}{2} \cdot h \\
&= \frac{x + (3\sqrt{2} + x + 3\sqrt{2})}{2} \cdot 3\sqrt{2} \\
\frac{30}{3\sqrt{2}} &= \frac{2x + 2 \cdot 3\sqrt{2}}{2} \\
5\sqrt{2} &= x + 3\sqrt{2} \\
x &= \boxed{2\sqrt{2}}
\end{aligned}$$