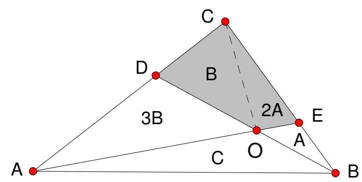
1. First draw $\overline{CO}$. Using area ratios, and letting $[BEO] = A, [CDO] = B$, we can get $[ADO] = 3B, [CEO] = 2A$. We don't know $[ABO]$ yet so let it be C.



Using the ratio $AO : OE$, we get the following proportion to solve for C in terms of A, B:

$$\frac{3B+B}{2A} = \frac{C}{A}$$
$$2C = 4B$$
$$C = 2B$$

Now we can use $AD : CD = 3 : 1$ to get the following proportions:

$$\frac{2B+3B}{A+2A+B} = \frac{3}{1}$$
$$\frac{5B}{3A+B} = 3$$
$$5B = 9A + 3B$$
$$2B = 9A$$
$$A = \frac{2B}{9}$$

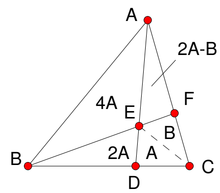
This now tells us that the total area of $\triangle ABC$ is equal to $2\left(\frac{2B}{9}\right) + \frac{2B}{9} + 2B + 3B + B = \frac{6B}{9} + 6B = \frac{2B}{3} + \frac{18B}{3} = \frac{20B}{3}$. Set this equal to 60:

$$\frac{20B}{3} = 60$$
$$20B = 180$$
$$B = 9$$

Now we know that $[DOEC] = 9 + 2(2) = \boxed{13}$.

2. This problem may have been graded incorrectly.

   First draw $\overline{EC}$. Using area ratios, and letting $[DCE] = A, [CEF] = B$, we can get $[BDE] = 2A, [ABE] = 4A, [AEF] = 2A - B$.



   We can then use $BE : EF$ to get the following proportions to be equal:

$$\frac{4A}{2A-B} = \frac{3A}{B}$$
$$4AB = 3A(2A - B)$$
$$4B = 6A - 3B$$
$$7B = 6A$$
$$A = \frac{7B}{6}$$

   Then we can plug our result into $2A - B = 2\left(\frac{7B}{6}\right) - B = \frac{4B}{3}$. This gives us the ratio of $AF : FC = \frac{4}{3} : 1 = 4 : 3$. Use this proportion to give us $AF = \boxed{\frac{28}{3}}$

3. In $\triangle ABC$, since $DC = 3BD$, we can say $[ABD] = \frac{1}{3}[ADC]$, or $[ABD] = \frac{1}{4}$ and $[ADC] = \frac{3}{4}$. By the same logic, since E is the midpoint of AD, then $[AEC] = [ECD] = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. We can also see that $[AEF] = [DEF]$. Note that $[BDF] = \frac{1}{7}$. Then $[AFD] = \frac{1}{4} - \frac{1}{7} = \frac{3}{28}$, and $[AEF] = \frac{1}{2} \cdot \frac{3}{28} = \frac{3}{56}$. So $[AEF] + [DEC] = \frac{3}{56} + \frac{3}{8} = \boxed{\frac{3}{7}}$

4. Since $\frac{BD}{DC} = \frac{1}{2}$, then $[ABD] = \frac{1}{3}$ and $[ADC] = \frac{2}{3}$. Since $[ABDE] = \frac{4}{5}$, then $[AED] = \frac{4}{5} - \frac{1}{3} = \frac{7}{15}$. Now we can find the ratio $\frac{AE}{EC}$. Because the triangles share the same height, the ratio of the bases will be the ratio of the areas. So $\frac{AE}{EC} = \frac{[AED]}{[CED]} = \frac{\frac{7}{15}}{\frac{1}{5}} = \frac{7}{15} \cdot \frac{5}{1} = \boxed{\frac{7}{3}}$

5. $\boxed{12}$