



Math Olympiad and Problem Solving Programs  
E230 - Advanced Math Competitions  
Problem Set 8.1 - Remainders

Name:

Date:

- Let's consider numbers of the form  $7k + 5$ , or multiples of 7 plus 5: 12, 19, 26, 33, 40, 47, 54, 61, 68, 75, 82, 89, 96. Now let's consider only the numbers in this list whose digits add up to 5, 10, or 15: 19, 82, 96. The sum of these numbers is  $19 + 82 + 96 = \boxed{197}$
- First, we need to understand that no matter where we start on the circle, the pattern of erasings will be the same. So let's draw the circle and start erasing from 1. If we follow the pattern, we will get that 3 is left. So if 1 is the first erased, 3 is the last. The pattern will be preserved throughout the circle; so if we start with 2, 4 will be last, if we start with 3, 5 will be last... So since 11 is last, we first erased  $\boxed{9}$ .
- $\boxed{39}$
- Let's consider all the possible values of  $k$ . Since  $1 < k < 8$ ,  $k$  can be 2, 3, 4, 5, 6, or 7.  $k$  is the divisor, so let's consider how many values of  $r$  will produce an integer for each  $k$ . If  $k = 2$ ,  $r = -2, 0, 2, 4$ , so there are 4 cases. If  $k = 3$ ,  $r = 0, 3$ ; 2 cases. For  $k = 4$ ,  $r = 0, 4$ ; 2 cases. For  $k = 5$ ,  $r = 0, 5$ ; 2 cases. If  $k = 6$  or 7, then  $r = 0$ , so there is 1 case for each of these. Count the cases:  $4 + 2 + 2 + 2 + 1 + 1 = 12$ . Count how many pairings of  $r \div k$ :  $8 \times 6 = 48$ . So divide  $\frac{12}{48} = \boxed{\frac{1}{4}}$
- The year is 697 days long, which is  $697 = 12 \times 58 + 1$ , so there are 58 weeks with 12 days in each week, plus 1 day left over. Using logic, we determine that the last day of the year falls on the same day of the week as the first day of the year. So if the first day of Year 0 falls on Day 1, the end of Y0 falls on D1, and the first day (and last day) of Y1 falls on D2, and the first and last day of Y2 falls on D3, ... Follow the pattern, and we determine that the first day of year 12 falls on D1.  $\boxed{12}$
- $\boxed{1}$
- $\boxed{600}$
- $\boxed{22}$
- We know that the 3-digit dividend ends in 2, and when divided it has a remainder of 1. Call the dividend  $d$ , the divisor  $k$ , and the quotient  $q$ . We can write the problem in this algebraic expression:  $\frac{d}{k} = q + \frac{1}{k}$ . Let's rearrange this equation a little. Multiply both sides by  $k$ , so we get  $d = kq + 1$ , and subtract 1 from both sides, so we get  $d - 1 = kq$ . Since  $d$ 's unit digit is 2,  $d - 1$  has a units digit of 1. So we need two integers  $k$  and  $q$  that multiply together to have a units digit 1. We can do this in three ways:  $1 \times 1, 3 \times 7, 9 \times 9$ . We can't use  $1 \times 1$  because then the divisor would be 1, and we would have no remainder. So we can only use  $3 \times 7$  and  $9 \times 9$ .  
Case 1:  $k = 3$ , so  $q$  ends in 7. The possible values of  $q = 37, 47, 57, \dots, 327$  and the values of  $kq = d - 1 = 111, 141, 171, \dots, 981$ . There are 30 numbers in this list.  
Case 2:  $k = 7$ , so  $q$  ends in 3. The possible values of  $q = 23, 33, 43, \dots, 133$  and the values of  $kq = d - 1 = 161, 231, 301, \dots, 931$ . There are 12 numbers in this list.  
Case 3:  $k = 9$ , so  $q$  ends in 9. The possible values of  $q = 19, 29, 39, \dots, 109$  and the values of  $kq = d - 1 = 171, 261, 351, \dots, 981$ . There are 10 numbers in this list.



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Now we must consider over counting. Notice all the values of  $d-l$  in case 3 are included in case 1. So we omit all of those numbers. Now we must consider where numbers in case 2 repeat numbers from case 1. The values of  $d-l$  in case 1 are of the form  $(d-1) = 111 + 30(n-1)$ . Manipulating this equation, we get  $\frac{(d-1) - 111}{30} + 1 = n$ . Since  $n$  is an integer, we will test the values of  $d-1$  in case 2 with this equation. If the solution is an integer, then we have a repeat. The values of  $d-1$  for case 2 that have integer solutions for the above equation are 231, 441, 651, and 861. So there are 4 repeats in case 2 that we must omit.

The final count is  $30 + 12 + 10 = 52$ , minus the  $10 + 4 = 14$  repeats, so there are  $52 - 14 = \boxed{38}$  possible dividends.

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