

1. (a) $\boxed{7}$ (b) $\boxed{1}$ (c) $\boxed{6}$

2. (a) $\boxed{0}$ (b) $\boxed{4}$

3. (a) $\boxed{x = 8}$ (b) $\boxed{x = 2}$

4. (a) $\boxed{5}$ (b) $\boxed{3}$ (c) $\boxed{2}$

(d) $(-10) \times (-9) \times (-8) \pmod{11} \equiv 1 \times 2 \times 3 \equiv \boxed{6}$

5. Solve the following equations

(a) $x + 9 \equiv 3 \pmod{11} \Rightarrow x \equiv -6 \pmod{11} \Rightarrow \boxed{x \equiv 5 \pmod{11}}$

(b) $3x + 7 \equiv 4 \pmod{11} \Rightarrow 3x \equiv -3 \pmod{11} \equiv 8 \pmod{11}$. This works for $x = 10$, because then we have $3x = 30 = 22 + 8$, or $8 \pmod{11}$. So $\boxed{x \equiv 10 \pmod{11}}$.

(c) $2 \equiv 5x - 4 \pmod{11} \Rightarrow -5x \equiv -6 \pmod{11} \Rightarrow 6x \equiv 5 \pmod{11}$. This first works for $x = 10$, because then we have $6x = 60 = 55 + 5$, or $5 \pmod{11}$. So $\boxed{x \equiv 10 \pmod{11}}$

6. Remember, an inverse of a number n is a number m such that $n \times m \equiv 1 \pmod{11}$. We use trial and error to find that

$$1 \times 1 \equiv 1 \pmod{11} \quad 2 \times 6 \equiv 12 \equiv 1 \pmod{11} \quad 3 \times 4 \equiv 12 \equiv 1 \pmod{11}$$

$$5 \times 9 \equiv 45 \equiv 1 \pmod{11} \quad 7 \times 8 \equiv 56 \equiv 1 \pmod{11} \quad 10 \times 10 \equiv 100 \equiv 1 \pmod{11}$$

So the following pairs are the inverses of each other: $\boxed{(1, 1), (2, 6), (3, 4), (5, 9), (7, 8), (10, 10)}$

7. First, let's make a chart of the mods of square numbers.

x	1	2	3	4	5	6	7	8	9	10
x^2	1	4	9	16	25	36	49	64	81	100
$\pmod{11}$	1	4	9	5	3	3	5	9	4	1

The steps for solving these problems are as follows:

(i) get the equation in positive form.

(ii) complete the square. We complete the square by taking the term multiplied on the x factor, divide by 2, square the resulting number, and add the resulting number to both sides.

(iii) use the square number chart to find the solutions.

(a) Since $-7x \equiv 4x \pmod{11}$, write the equation as $x^2 + 4x + 3 \equiv 0 \pmod{11}$. Now we need to complete the square. The number multiplied on the x term is 4, divided by 2 is 2, squared is 4. Since we already have 3, we just need to add one to both sides of the equivalence to have 4. So now we have $(x^2 + 4x + 3) + 1 \equiv (0) + 1 \pmod{11}$, or $x^2 + 4x + 4 \equiv 1 \pmod{11}$. Now the polynomial factors to $(x + 2)^2$, so we have $(x + 2)^2 \equiv 1 \pmod{11}$. We know from our square number chart that *something*² $\equiv 1 \pmod{11}$ if the *something* is 1 or 10. So $x + 2 \equiv 1 \pmod{11}$ or $x + 2 \equiv 10 \pmod{11}$. The first solves to $x \equiv -1 \equiv 10 \pmod{11}$, and the second solves to $x \equiv 8 \pmod{11}$. So $\boxed{x \equiv 8, 10 \pmod{11}}$



Math Olympiad and Problem Solving Programs
E220 - Intermediate Math Competitions
Problem Set 27.1 - Modular Operations

Name:

Date:

(b) $x^2 + 7 \equiv 0 \pmod{11} \Rightarrow x^2 \equiv -7 \pmod{11} \equiv 4 \pmod{11}$. A squared thing is equivalent to 4 for $x = 2$ and $x = 9$. So $x \equiv 2, 9 \pmod{11}$

(c) Since $-6x \equiv 5x \pmod{11}$, write the equation as $3x^2 + 5x + 2 \equiv 0 \pmod{11}$. Now we need to get rid of the 3 on the x^2 term, so we multiply both sides by the inverse of 3, which from problem 6 is 4. The equation becomes $4(3x^2 + 5x + 2) \equiv 0 \cdot 4 \pmod{11} \equiv 12x^2 + 20x + 8 \equiv x^2 + 9x + 8$. Now we complete the square. Let's change the $9x$ to $-2x$ for an easier time completing the square. Divide -2 by 2, which is -1 , and square it, which is 1. So we need 1 to complete the square, so rewrite the problem as follows: $x^2 - 2x + 8 \equiv 0 \pmod{11} \equiv x^2 - 2x + 1 + 7 \Rightarrow x^2 - 2x + 1 \equiv -7 \pmod{11} \equiv 4 \pmod{11}$. Then $x^2 - 2x + 1$ factors to $(x - 1)^2$, and the squared things that are equivalent to 4 are 2 and 9. So we have $x - 1 \equiv 2 \pmod{11}$ and $x - 1 \equiv 9 \pmod{11}$, which gives us $x \equiv 3, 10 \pmod{11}$

(d) Since the product of these three polynomials is 0, then one of them must be zero. This is possible if $2x + 1 \equiv 0 \pmod{11}$, $3x - 1 \equiv 0 \pmod{11}$, or $7x + 2 \equiv 0 \pmod{11}$. Solve these three equations:

$$2x + 1 \equiv 0 \pmod{11} \rightarrow 2x \equiv -1 \equiv 10 \pmod{11} \rightarrow x \equiv 5 \pmod{11}$$

$$3x - 1 \equiv 0 \pmod{11} \rightarrow 3x \equiv 1 \pmod{11} \rightarrow x \equiv 4 \pmod{11}$$

$$7x + 2 \equiv 0 \pmod{11} \rightarrow 7x \equiv -2 \equiv 9 \pmod{11} \rightarrow x \equiv 6 \pmod{11}$$

So the solution is $x \equiv 4, 5, 6 \pmod{11}$

8. 0

9. Altogether they have $83 + 129 = 212$ quarters and $159 + 266 = 425$ dimes. Now we need to find what coins will be left over after they put them in rolls, and we will use mods. $212 \equiv 12 \pmod{40}$ (we use mod 40 because the quarter rolls contain 40 quarters), and $425 \equiv 25 \pmod{50}$. So there are 12 quarters left and 25 dimes. The quarters are worth $12 \times 0.25 = \$3.00$, and the dimes are worth $25 \times 0.10 = \$2.50$, so the total worth of the leftover coins that Esther takes is $\$5.50$.