



Math Olympiad and Problem Solving Programs
E220 - Intermediate Math Competitions
Problem Set 4.1 - Counting Techniques

Name:

Date:

1. **18**. The restriction here is that the first digit cannot have 0. Thus, we have $3 \times 3 \times 2 \times 1 = 18$ numbers.
2. **15**. Let the 4 colors be B, R, Y, G, we can list the outcomes: $\{B, R, Y, G, BR, BY, BG, RY, RG, YG, BRY, BRG, BYG, RYG, BRYG\}$ for a total of 15 selections.
3. **19**. There are a total of 20 digits of 6 in the first 100 numbers. However, the number 66 contains 2 of them. Removing the duplication, we have 19 digit of 6.
4. **3498**. If no teacher is to have more than 8 students, the possible distribution of students is (8, 4), (7, 5), (6, 6), (5, 7), (4, 8). The case of (8, 4) and (4, 8) are the same. So do (7, 5) and (5, 7). Now, we focus on how to place students in Ms. Batty's class. Once the students in Ms. Batty's class are selected, the remaining students will automatically be assigned to Mr. Mirus's class with no additional selections needed.

For case (8, 4), we select 8 students from a total of 12. The permutation is P_8^{12} . However, since the order of selection does not matter, we have a total of $8!$ duplications. Thus, the final counts is $\frac{P_8^{12}}{8!}$.

With the same argument, we have $\frac{P_7^{12}}{7!}$ ways to select 7 students and $\frac{P_6^{12}}{6!}$ ways to select 6 students. Put it all together, we have $2\frac{P_8^{12}}{8!} + 2\frac{P_7^{12}}{7!} + \frac{P_6^{12}}{6!} = 3498$.

Remark: Selecting r objects from n candidates is also called combination, which is defined as $\binom{n}{r} = \frac{P_r^n}{r!}$. Therefore, we can simplify our solution as $2\binom{12}{8} + 2\binom{12}{7} + \binom{12}{6}$.

5. **959**. This problem looks easy, but very hard to get it right if you do not organize your work. There are many opportunities in making careless mistakes. The main one is fail to recognize numbers such as 64 are counted both as a perfect square 8^2 and a perfect cube 4^3 . Our basic strategy is to count the perfect powers and subtract it from 998 (Did you notice the trap that 1 and 1000 should not be counted?)

For perfect square, we have 2^2 to 31^2 for a total of 30 perfect squares.

For perfect cube, we have 2^3 to 9^2 for a total of 8 perfect cubes. However, $4^3 = (2^3)^2$ and $9^3 = (3^3)^2$ are also perfect squares counted above. Thus, correct for overcounting, we have 6 perfect cubes not counted before.

For perfect 4th power, since any 4th power can be expressed as square of squares, they are all counted in perfect square list.

For perfect 5th power, we have 2^5 and 3^5 for a total of 2 cases.

For perfect 6th power, we have $n^6 = (n^3)^2$ so it is also a perfect square. No cases.

For perfect 7th power, we have only 2^7 .

For perfect 8th and 9th power, we have 0 case.



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For perfect 10th power, we have $2^{10} = 1024$, which exceeded our range.

Therefore, we have a total of 39 cases. Therefore, we have $998 - 39 = 959$ numbers in the list.

6. The wording of this question made it difficult to solve with the skills taught in the class. The question is thrown out.
7. $\boxed{480}$. We use complimentary counting. There are $6!$ ways to line up the students with no restriction. If two girls are sit together, we have $5! \times 2!$ ways. Therefore, the number of ways the girls are not lining up together is $6! - 5! \times 2 = 480$.
8. $\boxed{144}$. There are two restrictions: the first digit cannot be 0 and last digit must be odd. There are 3 ways to select the last digit, 4 ways to select the first digit and 4×3 ways to select the remaining two digits. Thus, the number of selections is $4 \times 4 \times 3 \times 3 = 144$.
9. $\boxed{240}$. The driver is the restriction. There are 2 ways to select a driver and $5!$ ways to seat the remaining 5 people. Thus, we have $2 \times 5! = 240$ ways.
10. $\boxed{72}$. This is counting with restrictions. There are two registrations. The first is that the thousandth digit must be larger or equal to 6. There are only 2 digits meet the restriction (6 and 8). The unit digit must be an odd number. So, there are 3 choices (1, 3, 5). The remaining digit can only be used once so there are $2 \times 4 \times 3 \times 3 = 72$ numbers.