Math Olympiad and Problem Solving Programs
E210 - Introductory Math Competitions
Problem Set 22.1 - Casework
Name: Date:

$$\text{Permutation Formula: } P_r^n = \frac{n!}{(n-r)!}$$

Recall, $0! \equiv 1$.

1. (a) $P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = \boxed{360}$
   (b) $P_3^{10} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = \boxed{720}$
   (c) $P_1^{12} = \frac{12!}{(12-1)!} = \frac{12!}{11!} = \frac{12 \cdot 11!}{11!} = \boxed{12}$
   (d) $P_0^{9} = \frac{9!}{(9-0)!} = \frac{9!}{9!} = \boxed{1}$

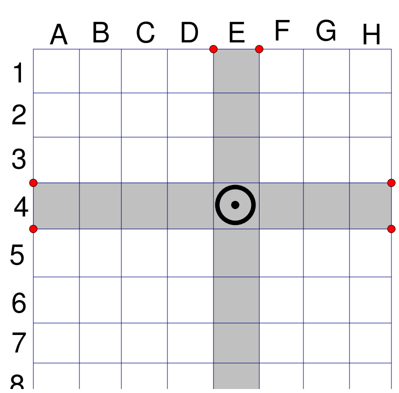
2. Whole numbers less than 100 are two-digit or one-digit numbers. There are 4 one-digit numbers that can be formed with the digits 1, 2, 3, and 4. There are $4 \times 3 = 12$ two-digit numbers that can be formed with these digits. So there are $4 + 12 = \boxed{16}$ numbers. (note: the answer key was wrong. If you would like your point back, please see the TA).

3. We start with the form _ _ _ _ and put in the blanks how many numbers can occupy each position. We need the number to be odd, so the last digit has to be odd. That means the last digit can be 1, 3, or 5. That means there are three choices for the last digit, so we can fill in _ _ _ $\underline{3}$. Now there are 5 digits left to choose from, so we write $\underline{5}$ _ _ $\underline{3}$. Now there are 4 digits left to choose from, and then there will be 3, so we write $\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{3}$. Now we multiply and get $\boxed{180}$

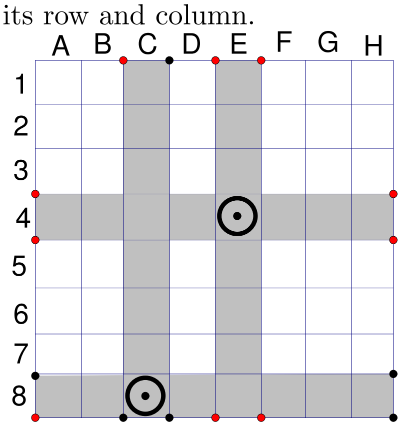
4. Again we start with the form _ _ _ _. There are 9 possible digits that can be put in the 1st place (we can't put 0 there or it would be a 3 digit number), so we can write $\underline{9}$ _ _ _. Then once we choose the first digit, we automatically know the last digit, because it's a palindrome so they must be the same. So we write $\underline{9}$ _ _ $\underline{1}$. Now we can choose the second digit. There are 10 choices for the digit, so we write $\underline{9}$ $\underline{10}$ _ $\underline{1}$, and we know that the third digit has to be the same as the second since it's a palindrome, so we write $\underline{9}$ $\underline{10}$ $\underline{1}$ $\underline{1}$. Multiply and we get $\boxed{90}$.

5. Rooks move in straight lines on the chessboard. So we need to make sure that none are in the same direct line from each other. Also, note that chessboards have labeled squares, so each square is unique on a chessboard.

   Let's place our first rook. There are 64 squares to choose from. Now cross out the row and column the rook is in.



Now there are 49 spaces to choose from on our chessboard. So select a random square for the second rook and cross out its row and column.



Now there are 36 squares available for the third rook. Continue placing rooks randomly and counting how many squares are available for placing rooks.

We find that there are $64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1$ combinations. Rewrite this: $8^2 \cdot 7^2 \cdot 6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2 \cdot 1^2 = (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^2 = \boxed{(8!)^2}$

6. Guess and check. If $m = 0$, then we have $(0,0), (0,1), (0,2)$, and $(0,3)$ that work. (because $0^2 + 0^2 < 10, 0^2 + 1^2 < 10$, etc).

   If $m = 1$, then we have $(1,0), (1,1), (1,2)$ that work.

   If $m = 2$, then we have $(2,0), (2,1), (2,2)$ that work.

   If $m = 3$, only $(3,0)$ works.

   So there are $\boxed{11}$ pairs.

7. We must use 1, 3, 5, 7, and 9 in our numbers less than 1000.

   There are 5 one-digit numbers.

   There are $5 \times 4 = 20$ two-digit numbers.

   There are $5 \times 4 \times 3 = 60$ three-digit numbers.

   So there are $5 + 20 + 60 = \boxed{85}$ such numbers.

8. $\boxed{70}$

9. So far, we can create $5 \times 3 \times 4 = 60$.

   Now let's trial and error adding 2 letters to each of the different sets, or adding 1 to two different sets.

   $(5 + 2) \times 3 \times 4 = 84$

   $5 \times (3 + 2) \times 4 = 100$

   $5 \times 3 \times (4 + 2) = 90$

   $(5 + 1) \times (3 + 1) \times 4 = 96$

   $(5 + 1) \times 3 \times (4 + 1) = 90$

$5 \times (3+1) \times (4+1) = 100$

So the maximum number of license plates is 100, and the maximum number of additional license plates is $\boxed{40 \text{ (D)}}$

10. Count them by different sizes:

    1×1: 6. 1×2: 4. 1×3: 2.

    2×1: 3. 2×2: 2. 2×3: 1.

    Now count the side-ways squares, and there are 2 of them. Add it all together and we get $\boxed{20}$ rectangles.